11 Incomplete Block Designs: Factorial Treatment Designs

The versatility of factorial treatment designs is explored in this chapter and the next with emphasis on 2^n and 3^n factorials. The effects and notations specific to factorial treatments are used to develop methods to construct incomplete block designs for factorial treatment designs in this chapter.

11.1 Taking Greater Advantage of Factorial Treatment Designs

The factorial treatment design was discussed in Chapters 6 and 7 as a means for investigating the effects of several treatment factors in the same experiment. The primary advantage of the factorial arrangement resides in the ability to determine whether the factors act independently or interact with one another as they affect the experimental units.

The 2^n and 3^n factorial treatment designs are of great practical importance and widely used in research studies. The 2^n factorials have n factors at two levels, and the 3^n factorials have n factors at three levels. As the number of factors increases, the number of treatment combinations rapidly escalates so that incomplete block designs are required to control experimental error. The factorial arrangement in each of these designs may be exploited to provide effective incomplete block designs for the investigation of factor effects and to facilitate the analysis of factor effects.

11.2 2^n Factorials to Evaluate Many Factors

The requirements of a research program may demand the investigation of many factors and their interrelationships as they affect the outcome of a process. Consider a large continuous production process in a chemical manufacturing plant that may not have as high a percent yield of the final product as projected by the plant construction specifications. The chemical yield of the final product is affected by factors at each of several separate reaction steps in the process. The problem is to identify factors that affect the final chemical yield and to determine the level of those factors that optimize the yield. The factors can include such items as the concentration of catalysts, the concentrations of reactants in solvents, and the ratio of one reactant to another, as well as temperature, pressure, agitation rates, and residence time in the reaction chambers.

Clearly, the engineers who are attempting to improve the plant performance have a formidable task before them. They have many factors to investigate with the potential that only a few are of major importance. They must screen the factors to determine the ones that warrant more detailed study, and at the same time they must control the cost of the experiments.

The 2^n factorial with many factors, each at a "low" level and a "high" level, can be used to detect the important factors in the process with a minimum of experimental units. Major trends can be detected with factors at two levels to identify potentially important factors. Consequently, 2^n factorials are often used at the early stage of experimentation to detect potential candidate factors for more detailed investigation. The following example with three factors illustrates the features of 2^n factorial treatment designs.

Example 11.1 Truck Leaf Spring Manufacture

An experiment described by Pignatiello and Ramberg (1985) was designed to investigate the effects of factors on a manufacturing process for leaf springs used on trucks. The assembled leaf spring was passed through a high-temperature furnace. Afterward, it was put into a forming machine to induce curvature in the spring by holding the spring in a high-pressure press for a short length of time.

The factorial treatment design for the experiment consisted of three factors, each at two levels. They were furnace temperature (A), heating time in the furnace (B), and transfer time between the furnace and the forming machine (C). The eight treatment combinations for the three factors in a 2^3 factorial are shown in Table 11.1, along with the observed measure of product quality (y) for one replication of each treatment.

New Treatment Labels

In general, the low level of a quantitative factor is denoted by a "0" and the high level by a "1." Equivalently, the two categories of a qualitative factor can be coded

Table 11.1 Truck leaf spring quality observations from a 2^3 factorial experiment

A Furnace Temperature (°F)	*B* Heating Time (sec)	*C* Transfer Time (sec)	*y*
1840	23	10	32
1880	23	10	35
1840	25	10	28
1880	25	10	31
1840	23	12	48
1880	23	12	39
1840	25	12	28
1880	25	12	29

as "0" and "1." Another useful notation for treatment combinations in 2^n factorials is illustrated with a 2^3 factorial in Display 11.1.

The uppercase letters A, B, and C represent factors. The "Treatment" label uses corresponding lowercase letters $a, b,$ and c. The lowercase letter is present if the factor is at level 1. The lowercase letter is absent if the factor is at level 0. The Treatment label is "(1)" if all factors are at level 0. The correspondence between the Treatment labels and the (0, 1) designations for factor levels are shown in Display 11.1, along with the actual levels of each factor in the truck spring experiment.

Evaluating the 2^n Factorial Effects

The *effect* of a factor with 2^n factorials corresponds to a change in the response from the low level to the high level of the factor. The simple effects, main effects, and interaction effects for factorials were discussed in Chapter 6. For illustration,

Display 11.1 Treatment Labels for the Truck Spring Experiment

Treatment	*A*	*B*	*C*	*Furnace Temperature (°F)*	*Heating Time (sec)*	*Transfer Time (sec)*	*y*
(1)	0	0	0	1840	23	10	32
a	1	0	0	1880	23	10	35
b	0	1	0	1840	25	10	28
ab	1	1	0	1880	25	10	31
c	0	0	1	1840	23	12	48
ac	1	0	1	1880	23	12	39
bc	0	1	1	1840	25	12	28
abc	1	1	1	1880	25	12	29

consider the observations from the truck spring experiment. The simple effects of Furnace Temperature as it changes from 1840° to 1880° F with factors Heating Time and Transfer Time held constant are shown in Table 11.2.

Table 11.2 Simple effects of Furnace Temperature at constant levels of Heating Time and Transfer Time for the truck spring experiment

Heating Time (B)	*Transfer Time (C)*	*Furnace Temperature (A)* 1840	1880	*Simple Effect*	*B*	*C*
23	10	32	35	35 − 32 = 3	0	0
25	10	28	31	31 − 28 = 3	1	0
23	12	48	39	39 − 48 = − 9	0	1
25	12	28	29	29 − 28 = 1	1	1

The main effect of a factor is the average effect of moving from its 0 level to its 1 level. Thus, the main effect of Furnace Temperature is the average of its simple effects

$$A = \frac{1}{4}[3 + 3 + (-9) + 1] = -0.5 \qquad \textbf{(11.1)}$$

Equivalent calculations produce the main effects $B = -9.5$ and $C = 4.5$.

Two factors, say A and B, interact if the effect of A is different at the two levels of B. When Heating Time is 23 seconds, $B = 0$ and the effect of Furnace Temperature is

$$(A \mid B = 0) = \frac{1}{2}[3 + (-9)] = -3$$

However, when Heating Time is 25 seconds, $B = 1$ and the effect of Furnace Temperature is

$$(A \mid B = 1) = \frac{1}{2}(3 + 1) = 2$$

If the furnace temperature is increased from 1840° to 1880° F the product quality is *reduced* three units if the heating time is 23 seconds. However, product quality is *increased* by two units if the heating time is 25 seconds. The response to the furnace temperature is different for the two heating times, implying the potential for an interaction between Furnace Temperature and Heating Time.

The interaction between two factors, say A and B, is defined as one-half the difference between the effect of A at $B = 1$ and $B = 0$. The estimate of interaction between Furnace Temperature and Heating Time is then

$$AB = \frac{1}{2}\left\{(A \mid B = 1) - (A \mid B = 0)\right\} = \frac{1}{2}[2 - (-3)] = 2.5$$

If the roles of factors A and B are interchanged the resulting interaction value is unchanged. Calculations for the other two-factor interactions give $AC = -3.5$ and $BC = -5.5$.

The three-factor interaction ABC arises if a two-factor interaction, say AB, is different for $C = 0$ and $C = 1$. The AB interaction is one-half the difference between the effect of A at $B = 1$ and $B = 0$. When transfer time is 10 seconds, $C = 0$ and the estimate of interaction between Furnace Temperature and Heating Time is

$$(AB \mid C = 0) = \frac{1}{2}(3 - 3) = 0$$

The response to A is three units regardless of the level for B. The zero value for AB interaction when $C = 0$ is represented graphically as two parallel response lines in Figure 11.1a.

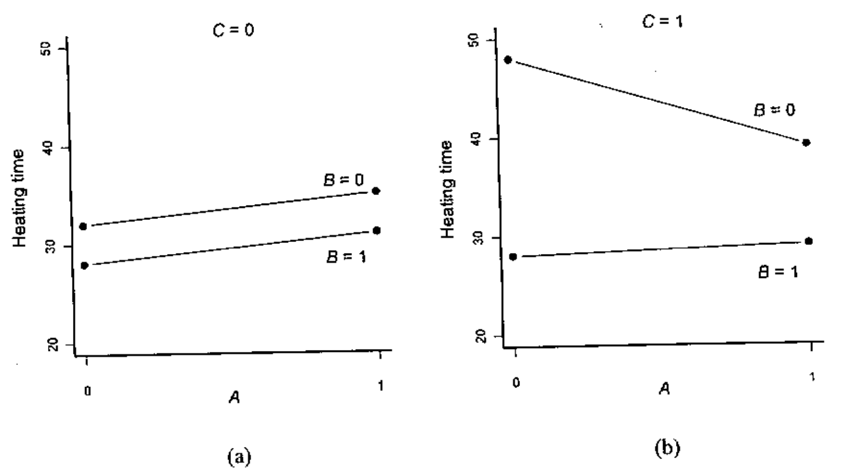

Figure 11.1 Graphic representation of ABC interaction depicted by different AB interaction at $C = 0$ and $C = 1$

When transfer time is 12 seconds, $C = 1$ and the interaction between Furnace Temperature and Heating Time is

$$(AB \mid C = 1) = \frac{1}{2}[1 - (-9)] = 5$$

The response to A is different for $B = 0$ and $B = 1$. The positive measure of AB interaction when $C = 1$ is represented as two different response lines in Figure 11.1b.

The interaction between Furnace Temperature and Heating Time differs with the transfer time. The estimate of AB interaction is 0 when transfer time is 10 seconds, and the estimate is 5 when the transfer time is 12 seconds.

One-half the difference between these two evaluations of the AB interaction is the three-factor interaction. The three-factor interaction estimate is

$$ABC = \frac{1}{2}\left\{(AB \mid C = 1) - (AB \mid C = 0)\right\} = \frac{1}{2}(5 - 0) = 2.5$$

The existence of a three-factor interaction translates into the two different graphic representations of the AB interaction in Figure 11.1. The same value for ABC interaction is attained if the AC interaction is evaluated at $B = 0$ and $B = 1$ or the BC interaction is evaluated at $A = 0$ and $A = 1$.

A Table of Contrasts to Summarize the Effects

The effects for 2^n factorials can be defined as contrasts with coefficients of $+1$ or -1 for each of the treatment combinations. A table of $+$ and $-$ signs determines the correct coefficient for any treatment combination in the contrast, and the table provides a systematic method for calculating factorial effects.

The full complement of $+$ and $-$ signs required for the contrasts in the 2^3 factorial for the truck spring experiment is shown in Table 11.3. The table of $+$ and $-$ signs is generated in the following manner:

Table 11.3 Coefficients for contrasts in a 2^3 factorial treatment design

	Factorial Effects								
Treatment	I	A	B	C	AB	AC	BC	ABC	y
(1)	+	−	−	−	+	+	+	−	32
a	+	+	−	−	−	−	+	+	35
b	+	−	+	−	−	+	−	+	28
ab	+	+	+	−	+	−	−	−	31
c	+	−	−	+	+	−	−	+	48
ac	+	+	−	+	−	+	−	−	39
bc	+	−	+	+	−	−	+	−	28
abc	+	+	+	+	+	+	+	+	29
Divisor	8	4	4	4	4	4	4	4	
Effect	33.8	−0.5	−9.5	4.5	2.5	−3.5	−5.5	2.5	
SS		0.5	180.5	40.5	12.5	24.5	60.5	12.5	

- The column denoted I, containing all $+$ signs, is used to estimate the grand mean with a divisor of 2^n.
- The next three columns, labeled with factors A, B, and C, have the $+$ and $-$ signs in *standard order*. The standard order has factor levels arranged

such that column A has successive pairs of $-$ and $+$ signs. Column B has pairs of $-$ signs followed by pairs of $+$ signs. Column C has four $-$ signs followed by four $+$ signs. In general the kth column has 2^{k-1} of the $-$ signs followed by an equal number of $+$ signs.

- The coefficients for any two-factor interaction are obtained as a product of the columns of coefficients for the corresponding main effects. For example, the coefficients for the AB column are the products of the corresponding elements in the A column and the B column.
- The coefficients for the three-way interaction ABC are obtained from the product of the coefficients for any set of columns whose symbol product is equal to ABC. The coefficients for ABC can be obtained from any of the symbolic products $A \times B \times C$, $AB \times C$, $AC \times B$, or $BC \times A$.

Column I often is referred to as the *Identity* column. Each column, except I, has an equal number of $+$ and $-$ signs.

Estimates of effects in Table 11.3 are computed by multiplying the corresponding column sign by the response y in each row of the table, summing the products, and dividing the sum by the appropriate divisor. The calculation of the main effect for Furnace Temperature in Equation (11.1) is the result of using the $+$ and $-$ coefficients in column A in Table 11.3 for the corresponding value of the response y and dividing by 4, or

$$A = \frac{1}{4}(-32 + 35 - 28 + 31 - 48 + 39 - 28 + 29) = -0.5$$

Similarly, utilizing the $+$ and $-$ codes under column B the main effect of Heating Time is

$$B = \frac{1}{4}(-32 - 35 + 28 + 31 - 48 - 39 + 28 + 29) = -9.5$$

The AB interaction effect is calculated using the $+$ and $-$ signs under the AB column to give

$$AB = \frac{1}{4}(32 - 35 - 28 + 31 + 48 - 39 - 28 + 29) = 2.5$$

In general, the contrast among treatment means for any main effect or interaction is

$$l_{AB...} = \sum_i k_i \bar{y}_i \qquad (11.2)$$

where the coefficients for the contrasts are $k_i = \pm 1$. Following the convention for calculating contrasts, standard errors, and sums of squares in Chapter 3, the estimate of the effect for any contrast among treatment means in a complete 2^n factorial can be expressed as

$$AB... = \frac{1}{2^{n-1}}(l_{AB...}) \qquad (11.3)$$

The standard error estimate for an effect estimate is

$$s_{AB...} = \sqrt{\frac{4\sigma^2}{r2^n}} \qquad (11.4)$$

where r is the number of replications for each treatment. The 1 degree of freedom sum of squares for the effects shown at the bottom of Table 11.3 is obtained with

$$SS(\mathbf{AB}...) = \frac{r}{2^n}(l_{AB...})^2 \qquad (11.5)$$

For the example in Table 11.3, $r = 1$ because each treatment combination only occurs one time. The 1 degree of freedom sum of squares for A is

$$SS\mathbf{A} = \frac{1}{8}(-2)^2 = 0.5$$

while those for B and AB are

$$SS\mathbf{B} = \frac{1}{8}(-38)^2 = 180.5 \quad \text{and} \quad SS(\mathbf{AB}) = \frac{1}{8}(10)^2 = 12.5$$

11.3 Incomplete Block Designs for 2^n Factorials

The use of incomplete block designs to reduce experimental error variance was introduced in Chapter 9. The performance of a complete replication for the 2^n factorials with many factors may not be possible in a single complete block. If there is insufficient raw material in a manufactured batch to accommodate all of the treatments, each batch of raw materials can be used as an incomplete block. If experimental error is too large with complete block designs in agricultural field experiments, the variation among field plots can be controlled in the experiment with reduced block sizes for more homogeneous groups of experimental plots. Blocks of reduced size for 2^n factorials can be devised by exploiting the construction of effect contrasts in the 2^n factorials.

Sacrifice Treatment Information to Increase Precision

Incomplete block designs for 2^n factorials are constructed such that one or more treatment contrasts are identical to block contrasts. The treatment effect is said to be completely **confounded** with blocks. The confounded treatment effect is indistinguishable from the effect of the blocks with which it is confounded.

Ordinarily the highest order interaction effect in a 2^n factorial is chosen to be confounded with blocks. Main effects, two-factor interactions, and other lower order interactions, in the case of experiments with many factors, are those effects of most interest. By confounding the highest order interaction the other effects are estimated without penalty.

The construction of an incomplete block design is illustrated with a 2^3 factorial. A complete block design requires eight experimental units per block. Half of the treatments have a + coefficient and half have a − coefficient for every effect. A resolvable incomplete block design with two blocks of four units each per replication can be constructed using the contrast l_{ABC}.

From Table 11.3, the ABC interaction is estimated with the comparison

$$l_{ABC} = abc + a + b + c - ab - ac - bc - (1)$$

Put the treatment combinations with a + coefficient—abc, a, b, and c—in one block and the treatment combinations with a − coefficient—ab, ac, bc, and (1)—in the other block. The two incomplete blocks of treatments are shown in Display 11.2.

Display 11.2 *ABC* Interaction Confounded in Two Blocks of Four Experimental Units Each

	Block 1	Block 2
	abc	*ab*
	a	*ac*
	b	*bc*
	c	*(1)*
ABC	+1	−1

The comparison (block 1 − block 2) is the contrast required to estimate the ABC interaction. As a consequence, the estimate of the ABC interaction effect is completely confounded with a comparison between the blocks. It will not be possible to estimate the three-factor interaction independent of block effects.

On the other hand, the other six factorial effects are not confounded with blocks and can be estimated in the usual way. For example, the main effect of factor A estimated from

$$l_A = (abc + a - b - c) + (ab + ac - bc - (1))$$

contains two + coefficients and two − coefficients from the four units in each block. Any differences among the blocks will not affect the estimate.

Block Construction with the Evens–Odds Rule

Half of the treatments have a + coefficient and half have a − coefficient for every effect in the 2^n factorials. The treatments can be divided into the two groups with the *Evens–Odds* rule.

Any treatment combination that has an even number of letters from the factorial effect receives one of the coefficients (+ or −). Treatments with an odd number of letters from the factorial effect receive the other coefficient.

The ABC interaction has the associated letters a, b, and c. Treatments (1), ab, ac, and bc have an even number of letters from the ABC factorial effect. Treatment (1) with zero letters has an even number of letters, since zero is considered an even number. Treatments a, b, c, and abc have an odd number of letters. The treatment combinations with the + coefficient for the ABC interaction effect are

$$a,\ b,\ c, \text{ and } abc$$

and the treatment combinations with the − coefficient for the ABC interaction effect are

$$(1),\ ab,\ ac, \text{ and } bc$$

The treatment combination that contains all of the factor letters, abc in this case, will always have the + coefficient for any factorial effect. All of the treatment combinations in that group will also have the + coefficient. The treatment combinations in the other group will have the − coefficient for the factorial effect. This assignment of + and − coefficients for the contrast coincides with that shown in Table 11.3.

Suppose the $ABCD$ interaction from a 2^4 factorial is to be confounded with blocks for an experiment with two blocks of eight experimental units per block in each replication. The letters associated with the four-factor interaction effect $ABCD$ are a, b, c, and d. The treatment combinations with an even number of letters are

$$(1),\ ab,\ ac,\ ad,\ bc,\ bd,\ cd, \text{ and } abcd$$

and the treatment combinations with an odd number of letters are

$$a,\ b,\ c,\ d,\ abc,\ abd,\ acd, \text{ and } bcd$$

The treatments in the group containing the treatment combination $abcd$ receive a + coefficient since the treatment combination $abcd$ contains the letters of all four factors. The other group of treatments all have a − coefficient for the $ABCD$ interaction effect and are placed in a separate block from the first group.

Analysis of Variance Outline for a Completely Confounded Design

The sums of squares are computed as usual for the analysis of variance except for the exclusion of a sum of squares partition for the interaction effect confounded with blocks. The block sum of squares will include the confounded factorial effect.

The sources of variation and degrees of freedom for the analysis of variance are outlined for the 2^3 factorial with $b = 2$ incomplete blocks in each of $r = 2$ replicates in Table 11.4. Since ABC is confounded with blocks the sum of squares for blocks includes the ABC effect.

Table 11.4 Analysis of variance for a 2^3 factorial with $b = 2$ incomplete blocks in each of $r = 2$ replicate groups

Source of Variation	*Degrees of Freedom*	
Replicates	$r - 1 = 1$	
Blocks within replicates	$r(b - 1) = 2$	
Treatments	6	
A		1
B		1
C		1
AB		1
AC		1
BC		1
Error	6	
Total	15	

Retain Some Treatment Information with Partial Confounding

Block size was reduced in the previous section by confounding the highest order interaction with blocks. However, any gain that may occur with a reduced experimental error has its price. In the previous designs, the loss of total information on the confounded factorial effect was the cost for a possible reduction in the experimental error variance.

Any factorial effect can be confounded with blocks. To avoid the loss of all information on any one factorial effect a different effect can be confounded in each replication group of the resolvable design. In this way a factorial effect is only confounded in one of the replications and thus is said to be *partially confounded* with blocks.

A 2^3 factorial with a different effect confounded in each of three replications is used to illustrate the principle in Example 11.2.

Example 11.2 Partial Confounding in a 2^3 Factorial

The purity of a chemical product was thought to be influenced by three factors—agitation rate (A), base component concentration (B), and concentration of reagent (C). The chemist set up an experiment using a factorial treatment design with each of the factors at two levels for a 2^3 factorial arrangement.

Experiment Design: The chemist wanted three replications of the experiment, but only four runs of the chemical process could be conducted in a single day. Therefore, each replication had to be run in two incomplete blocks (days).

An effect contrast for a 2^3 factorial consists of four treatment combinations with a + coefficient and four treatment combinations with a − coefficient. Thus, it was possible to construct the incomplete block design by confounding a 2^3 factorial effect with blocks. To avoid complete confounding of one effect, the chemist confounded a different two-factor interaction in each replication. The treatment combinations required for each block are shown in the diagram in Table 11.5.

Table 11.5 Observed purity of a chemical product in a partially confounded 2^3 factorial

BC Confounded				*AC Confounded*				*AB Confounded*			
+1		−1		+1		−1		+1		−1	
(1)	25	*ab*	43	*abc*	39	*bc*	38	(1)	26	*a*	43
bc	34	*c*	30	*b*	29	*a*	37	*c*	32	*b*	34
abc	42	*ac*	40	(1)	27	*ab*	46	*ab*	52	*ac*	40
a	25	*b*	33	*ac*	40	*c*	34	*abc*	51	*bc*	36
Block 1		Block 2		Block 3		Block 4		Block 5		Block 6	
Replicate I				Replicate II				Replicate III			

The BC, AC, and AB interactions were each confounded in one replication of the experiment. The two-factor interaction confounded with the two blocks in each replication is shown above the blocks. The observed purity rates are to the right of the treatment combinations in each block.

The treatment combinations required in each of the incomplete blocks can be determined with the Evens–Odds rule. In replicate I the BC interaction is confounded by placing treatment combinations with an even number of the letters b and c together in block 1. They are (1), a, bc, and abc. The treatments with an odd number of the letters b and c are placed in block 2. They are c, b, ab, and ac. The treatments in block 1 have the +1 coefficient for the BC interaction effect because the treatment combination abc contains all of the factor letters. The treatments in block 2 have the −1 coefficient for the BC interaction effect. Consequently, the difference between the observations in block 1 and block 2 will have the estimate of the BC interaction effect confounded with the difference between the effects of blocks 1 and 2. The treatment combinations in each of the blocks of replicates II and III can be determined in a similar manner with the AC effect confounded in replicate II and the AB effect confounded in replicate III.

Computing the Sums of Squares for Partially Confounded Designs

The sums of squares for the effects not confounded with blocks, A, B, C, and ABC, may be calculated in the usual manner. The sums of squares can be calculated from Equation (11.5) using the coefficients for the contrasts in Table 11.3. The effects can be estimated in all replications, thus the sum of squares is $SS = r(\Sigma k_i \bar{y}_i)^2/2^n$, where $r = 3$, $2^n = 8$, and $k_i = \pm 1$. The calculations are shown in Table 11.6.

Table 11.6 Computations for factorial effects not confounded with blocks

	(1)	*a*	*b*	*ab*	*c*	*ac*	*bc*	*abc*			
Means:	26	35	32	47	32	40	36	44	$\Sigma k_i \bar{y}_i$	2^n	SS
A	−	+	−	+	−	+	−	+	40	8	600.00
B	−	−	+	+	−	−	+	+	26	8	253.50
C	−	−	−	−	+	+	+	+	12	8	54.00
ABC	−	+	+	−	+	−	−	+	− 6	8	13.50

The sum of squares for each of the partially confounded two-factor interactions must be calculated from the replications in which they are not confounded. The sum of squares for AB must be calculated from replicates I and II, for AC from replicates I and III, and for BC from replicates II and III. Totals are used instead of the means for these calculations.

The sum of squares can be computed conveniently by determining the effect contrast from all of the observations and subtracting the value of the contrast represented by the difference of block totals for the replications in which the effect is confounded. The three confounded contrasts, AB, AC and BC, computed from the totals of all treatment observations are

Treatments	(1)	*a*	*b*	*ab*	*c*	*ac*	*bc*	*abc*	
Totals	78	105	96	141	96	120	108	132	$\Sigma k_i y_i$
AB	+	−	−	+	+	−	−	+	18
AC	+	−	+	−	−	+	−	+	− 24
BC	+	+	−	−	−	−	+	+	− 30

The totals and difference computed for each pair of blocks are

	Replicate I *BC Confounded*		*Replicate II* *AC Confounded*		*Replicate III* *AB Confounded*	
	Block 1	*Block 2*	*Block 3*	*Block 4*	*Block 5*	*Block 6*
Totals	126	146	135	155	161	153
Difference	− 20		− 20		8	

To estimate the AB effect contrast only from replicates I and II, the difference between blocks 5 and 6 (8) is subtracted from the AB contrast obtained from the totals of all observations, 18. The corrected estimate of the contrast is

$$l_{AB} = \frac{18 - 8}{2} = 5$$

where the divisor is $r = 2$ to put the result on a mean basis. The sum of squares for AB interaction is

$$SS(\boldsymbol{AB}) = \frac{2(5)^2}{8} = 6.25$$

Similarly, the estimate of the AC contrast is

$$l_{AC} = \frac{-24 - (-20)}{2} = -2$$

with sum of squares

$$SS(\boldsymbol{AC}) = \frac{2(-2)^2}{8} = 1$$

Finally,

$$l_{BC} = \frac{-30 - (-20)}{2} = -5$$

and

$$SS(\boldsymbol{BC}) = \frac{2(-5)^2}{8} = 6.25$$

The analysis of variance is shown in Table 11.7. The design is a resolvable incomplete block design with blocks of treatments in complete replication groups. The total sum of squares and the sums of squares for replicates and for blocks within replicates are computed in the usual manner. The sum of squares for experimental error is found by subtracting the sums of squares for all treatment effects, replicates, and blocks within replicates from the total sum of squares.

Tests of Hypotheses About Factor Effects

The critical value for a test of hypothesis for any factorial effect is $F_{.05,1,11} = 4.84$. The agitation rate (A) and base component concentration (B) main effects are significant. None of the other effects had a significant effect on the purity of the chemical product.

From Equations (11.3) and (11.4), the estimated effect of agitation rate is

$$A = \frac{(40)}{4} = 10$$

Table 11.7 Analysis of variance for purity of chemical product in a partially confounded 2^3 factorial

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	F_0
Replicates	2	111.00	55.50	
Blocks within reps	3	108.00	36.00	
A	1	600.00	600.00	40.6
B	1	253.50	253.50	17.2
C	1	54.00	54.00	3.7
ABC	1	13.50	13.50	< 1
AB (Reps I, II)	1	6.25	6.25	< 1
AC (Reps I, III)	1	1.00	1.00	< 1
BC (Reps II, III)	1	6.25	6.25	< 1
Error	11	162.50	14.77	
Total	23	1,316.00		

with standard error

$$s_A = \sqrt{\frac{4(14.77)}{3(8)}} = 1.57$$

The estimated effect of the base component concentration is

$$B = \frac{26}{4} = 6.5$$

with standard error $s_B = 1.57$. The purity of the chemical product is increased by 10 units if the agitation rate is increased from the low level to the high level. The purity is increased by 6.5 units if the concentration of the base component is increased from the low level to the high level. The reagent concentration C had no significant effect on the purity of the product, and there was no significant interaction among any of the factors.

Confound Another Interaction to Further Reduce Block Size

The number of experimental units per block can be reduced further by choosing a second factorial effect to confound with blocks. The design will have four blocks per replication if two factorial effects are used for confounding.

The technique is illustrated with the 2^3 factorial to produce four blocks of two experimental units. If the interaction effects AB and AC are chosen as confounding effects, the assignment of $+$ and $-$ coefficients to the treatments by the Evens–Odds rule is

	(1)	abc	c	ab	b	ac	a	bc
AB	+	+	+	+	−	−	−	−
AC	+	+	−	−	+	+	−	−

Four blocks of two experimental units are formed by taking treatment pairs that have the same configurations of $+$ and $-$ coefficients for the AB and AC interactions.

There are four configurations of $+$ and $-$ coefficients for the joint AB and AC interaction contrasts. Each configuration contains two treatments. The two treatments are each assigned to a separate block. The configurations and treatment assignments to blocks are

AB	*AC*	*Treatment*		*Block*
+	+	(1)	abc	1
+	−	c	ab	2
−	+	b	ac	3
−	−	a	bc	4

A Third Effect Is Automatically Confounded

If two effects are confounded with blocks in a 2^n factorial, then a third effect is confounded. There are 3 degrees of freedom for block comparisons, and three treatment effects are confounded with blocks. The AB interaction is confounded with the block contrast

$$AB = l_1 = B_1 + B_2 - B_3 - B_4$$

where B_1, B_2, B_3, and B_4 represent block means. The AC interaction is confounded with the block contrast

$$AC = l_2 = B_1 - B_2 + B_3 - B_4$$

The third contrast among the blocks that is orthogonal to $AB = l_1$ and $AC = l_2$ is

$$l_3 = B_1 - B_2 - B_3 + B_4$$
$$= (1) + abc - c - ab - b - ac + a + bc$$

If the contrasts in Table 11.3 are checked it can be seen that the third contrast is equal in the BC interaction.

The Third Confounded Effect Is a Generalized Interaction

The third effect confounded is known as the **generalized interaction** of the first two confounded effects. The generalized interaction of two effects is obtained by forming the symbolic product of the two effects and striking out any letters that appear twice in the product.

For example, AB and AC were chosen as the confounding effects. Their symbolic product is $ABAC$. Striking out the letter A, which appears twice, the result is $ABAC = \not{A}B\not{A}C = BC$. The generalized interaction is BC. Suppose ABC and AB are chosen as confounding effects. The generalized interaction is $ABCAB = \not{A}\not{B}C\not{A}\not{B} = C$. The main effect C is confounded with blocks along with ABC and AB. Care must be taken to avoid confounding effects that are of particular interest in the study, especially main effects.

A table is given in Appendix 11A to aid in the construction of useful incomplete block designs for 2^n factorials. A general method of confounding that can be used with factorial systems other than the 2^n series is presented in the next section.

11.4 A General Method to Create Incomplete Blocks

The allocation of treatment combinations to incomplete blocks for the 2^n factorials has been accomplished thus far on the basis of the $\pm$ signs of the effects confounded with blocks. However, the method becomes rather cumbersome with many factors due to the large number of treatment combinations involved in defining the contrasts. In addition, the use of $\pm$ signs for the confounding system does not carry over to other systems such as the 3^n factorials.

A general method to construct incomplete block designs with chosen factorial effects confounded with blocks utilizes the mathematics of *residues modulo m* or *residues mod m*. For an integer k the residue mod m is the remainder when k is divided by m. The residue r for the integer k mod m is written as $k = r(\text{mod } m)$.

With 2^n factorials we work with residues of (mod 2). Any integer divided by $m = 2$ leaves a remainder of 0 or 1. Thus, the values for the residues (mod 2) are 0 and 1. The even integers (mod 2) have residue 0 and the odd integers (mod 2) have residue 1. For example, $7 = 1(\text{mod } 2)$ since 7 divided by 2 is 3 with a remainder of 1. Also, $4 = 0(\text{mod } 2)$ since 4 divided by 2 is 2 with a remainder of 0.

The levels of a factor (0 or 1) are used as the values of a variable x_i representing the ith factor. A treatment combination for the 2^n factorial is represented as the sequence $x_1x_2x_3 \cdots x_n$. For a 2^3 factorial the eight treatments written in standard order are

x_1	x_2	x_3	*Treatment*
0	0	0	(1)
1	0	0	*a*
0	1	0	*b*
1	1	0	*ab*
0	0	1	*c*
1	0	1	*ac*
0	1	1	*bc*
1	1	1	*abc*

A Defining Contrast for Two Blocks

A general method to determine the allocation of treatment contrasts to incomplete blocks is accomplished with a linear function

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \qquad (11.6)$$

where L is the *defining contrast*, or the contrast confounded with blocks. The value of α_i is 1 if the ith factor is present in the defining contrast and 0 if the factor is absent in the defining contrast. The defining contrast function L is evaluated for each treatment combination. The value of x_i is the level of the ith factor (0 or 1) in any treatment combination under consideration for allocation to an incomplete block.

Suppose the defining contrast for a 2^3 factorial in two blocks of four units each is the two-factor interaction AB. Factors A and B are present in the defining contrast so that $\alpha_1 = 1$ and $\alpha_2 = 1$. Since factor C is not in the defining contrast, $\alpha_3 = 0$. The defining contrast function for allocation of treatments is

$$L = x_1 + x_2 \qquad (11.7)$$

The values of x_1 and x_2 for each treatment combination are substituted into L and the residue for L modulo 2, $r(\text{mod } 2)$, is determined. For example, the value of $L = x_1 + x_2$ for the treatment combination bc, $x_1x_2x_3 = (011)$, is $L = 0 + 1$. The residue for $L = 1$ is $1(\text{mod } 2)$. The values of L and residues, $L = r(\text{mod } 2)$, are

Treatment	x_1	x_2	$L = x_1 + x_2$	$r(\text{mod } 2)$
(1)	0	0	0	0
a	1	0	1	1
b	0	1	1	1
ab	1	1	2	0
c	0	0	0	0
ac	1	0	1	1
bc	0	1	1	1
abc	1	1	2	0

The treatment combinations with $L = 0(\text{mod } 2)$ are assigned to one block, and the treatment combinations with $L = 1(\text{mod } 2)$ are assigned to the other block. Thus, the treatment assignments are

Block 1	(1) *ab* *c* *abc*	with $L = 0$ (mod 2)
Block 2	*a* *b* *ac* *bc*	with $L = 1$ (mod 2)

Let B_1 and B_2 represent the block totals. The block with residue 0 for L, block 1, contains the treatment combinations with a + sign for the AB contrast, while block 2 with residue 1 for L contains the treatment combinations with a $-$ sign for the AB contrast. Thus, the contrast l among block totals equivalent to the AB contrast is

$$l_{AB} = B_1 - B_2$$

Use Two Defining Contrasts for Four Blocks

The designs to this point have been constructed for 2^n factorials with two incomplete blocks of 2^{n-1} experimental units, each with one defining contrast confounded with blocks. The use of two blocks with one factorial effect confounded may not reduce block size sufficiently for a 2^n factorial when there are many factors, say $n \geq 4$. For example, a 2^5 factorial with 32 treatments may require block sizes no larger than eight units per block with four blocks per replication. Further reductions in block size can be accomplished by confounding an additional defining contrast with blocks.

Suppose a 2^4 factorial is to have the 16 treatment combinations placed in four blocks of $2^{4-2} = 2^2 = 4$ experimental units each. Two defining contrasts are required to construct the four incomplete blocks. If AB and CD are chosen to be confounded with blocks, then the defining contrasts are

$$L_1 = x_1 + x_2 \qquad (11.8)$$

$$L_2 = x_3 + x_4$$

where L_1 represents AB and L_2 represents CD. Each treatment combination will provide a pair of residues modulo 2 for the pair (L_1, L_2). There are four pairs of residues possible—(0,0), (0,1), (1,0), and (1,1). Treatment combinations with the same values for a pair of residues modulo 2 are placed in the same incomplete block. The values for the defining contrasts, L_1 and L_2 in Equation (11.8), and the residue pairs for each of the 16 treatment combinations are shown in Display 11.3 along with the block assignments.

The 3 degrees of freedom among the blocks represent three orthogonal contrasts among the blocks. Two of the contrasts are known to include the two-factor interactions, AB and CD, chosen to be confounded with blocks. The contrast for the AB interaction is the difference between the blocks with residues 0 and 1 for L_1. Let B_1, B_2, B_3, and B_4 represent the block totals. The block contrast for AB is

$$l_{AB} = B_1 + B_2 - B_3 - B_4$$

because blocks 1 and 2 have residue 0 and blocks 3 and 4 have residue 1 for L_1. Similarly, the blocks with residues 0 and 1 for L_2 define the contrast for the CD interaction. The block contrast for CD is

$$l_{CD} = B_1 - B_2 + B_3 - B_4$$

because blocks 1 and 3 have residue 0 and blocks 2 and 4 have residue 1 for L_2.

Display 11.3 Incomplete Block Design for a 2^4 Factorial in Four Blocks of Four Units with Defining Contrasts AB and CD

Block	*Treatment*	x_1	x_2	x_3	x_4	L_1	L_2	*Residue*
1	(1)	0	0	0	0	0	0	(0,0)
	ab	1	1	0	0	2	0	(0,0)
	cd	0	0	1	1	0	2	(0,0)
	abcd	1	1	1	1	2	2	(0,0)
2	*c*	0	0	1	0	0	1	(0,1)
	d	0	0	0	1	0	1	(0,1)
	abc	1	1	1	0	2	1	(0,1)
	abd	1	1	0	1	2	1	(0,1)
3	*a*	1	0	0	0	1	0	(1,0)
	b	0	1	0	0	1	0	(1,0)
	acd	1	0	1	1	1	2	(1,0)
	bcd	0	1	1	1	1	2	(1,0)
4	*ac*	1	0	1	0	1	1	(1,1)
	bc	0	1	1	0	1	1	(1,1)
	ad	1	0	0	1	1	1	(1,1)
	bd	0	1	0	1	1	1	(1,1)

$L_1 = x_1 + x_2 \quad L_2 = x_3 + x_4$

Determining the Generalized Interaction

The $ABCD$ interaction is the *generalized interaction* confounded as a consequence of purposely confounding AB and CD with blocks. The generalized interactions can be determined by more formal algebraic rules than those described in the previous section. Form the product of the symbols for the defining contrasts with the exponent of any symbol reduced modulo 2. The product of AB by CD is

$$AB \times CD = ABCD$$

The generalized interaction of AB and CD as determined by the symbol product is $ABCD$ since all exponents of the symbol product are 1(mod 2).

Suppose the contrasts ABC and BCD had been chosen for the original defining contrasts. The product is $ABC \times BCD = ABBCCD = AB^2C^2D = AD$, where B^2 and C^2 do not appear in the generalized interaction because their exponents are 0(mod 2). Therefore, the generalized interaction is AD when ABC and BCD are the defining contrasts.

Because main effects and two-factor interactions are of greatest interest in preliminary studies of factor effects, we try to avoid confounding them with blocks. The number of experimental units in an incomplete block design for 2^n factorials is equal to the powers of 2—2, 4, 8, and so forth—up to 2^{n-1}. In general, if blocks of size $k = 2^q$ are required there will be $2^n/2^q = 2^{n-q}$ blocks in a complete replication. Consequently, $n - q$ defining contrasts must be chosen.

Consider the 2^7factorial with 128 treatment combinations. A design with $2^4 = 16$ experimental units per block requires $2^{7-4} = 2^3 = 8$ blocks and $(7-4) = 3$ defining contrasts. Choose ABG, CDE, and EFG as the defining contrasts to reduce block sizes. The first contrast, ABG, is used to reduce block sizes to 64 units. The second, CDE, is used to reduce block sizes to 32 units, and the third contrast, EFG, is used to reduce block sizes to 16 units. There will be four generalized interactions also confounded with the eight blocks. They are

$$ABG \times CDE = ABCDEG$$

$$ABG \times EFG = ABEFG^2 = ABEF$$

$$CDE \times EFG = CDE^2FG = CDFG$$

and

$$ABG \times CDE \times EFG = ABCDE^2FG^2 = ABCDF$$

As before, the treatment combinations are assigned to the blocks according to the residues of the defining contrasts

$$\begin{array}{lcl} ABG & \longrightarrow & L_1 = x_1 + x_2 + x_7 \\ CDE & \longrightarrow & L_2 = x_3 + x_4 + x_5 \\ EFG & \longrightarrow & L_3 = x_5 + x_6 + x_7 \end{array}$$

Each treatment contrast will have a triplet of values (L_1, L_2, L_3). There will be eight triplets of residues—(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), and (1, 1, 1)—each representing the residue triplet for a block assignment of the treatment combinations. A treatment combination with the residue triplet (0, 0, 0) is assigned to one block, a treatment combination with the residue triplet (0, 0, 1) is assigned to a second block, and so forth.

Table 11A.1 in the Appendix lists defining contrasts and their generalized interactions to construct incomplete block designs for 2^n factorials with block sizes $k \geq 4$ for $n = 4, 5, 6$, and 7 factors. Notice that some two-factor interactions will be confounded with blocks when blocks have four or less units. Designs with only three-factor and larger interactions confounded are always possible with block sizes of eight or more. If the defining contrasts are properly chosen it will be possible to avoid confounding any two-factor interactions or main effects.

11.5 Incomplete Block Designs for 3^n Factorials

The 2^n factorials are useful designs to detect factors with major effects on the measured responses in an experiment. The 3^n factorials have three levels for each factor, making it possible to estimate linear and quadratic trends for quantitative factors and to provide more detailed descriptions of qualitative factor effects. However, the number of experimental units required by 3^n factorials increases by powers of 3 as more factors are added. Thus, incomplete block designs can be very useful with these treatment designs. The construction of incomplete blocks designs for 3^n factorials is discussed briefly in this section.

Some 3^n Basics

Notation for 3^n Factorials

The levels of a factor are represented by $x_i = 0, 1, 2$. For example, the nine treatment combinations for a 3^2 factorial with factors A and B are

		A		
		0	1	2
	0	00	10	20
B	1	01	11	21
	2	02	12	22

The three columns of the array represent treatment combinations for the three levels of factor A, and the three rows represent treatment combinations for the three levels of factor B.

Three Incomplete Blocks Required for Confounded 3^n Factorials

The construction of incomplete block designs for 3^n factorials requires three blocks to have blocks of equal size. There will be 2 degrees of freedom between blocks, and a treatment effect with 2 degrees of freedom must be confounded with blocks.

The 3^n factorials have 2 degrees of freedom for main effects, 2^2 degrees of freedom for two-factor interactions, and so forth. We would not want to confound the main effects with blocks. Instead, the interactions are partitioned into two orthogonal components, each of which has 2 degrees of freedom. The orthogonal component can then be used as the defining contrast in the construction of the incomplete block design.

Construct Incomplete Blocks with Defining Contrasts

The allocation of treatment combinations to blocks utilizes the defining contrast function $L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ introduced in Section 11.4. The value of x_i for the 3^n factorial is $x_i = 0, 1, 2$. As a mathematical convenience, the values of

$\alpha_i = 0, 1, 2$ are used for factors included in the 2 degrees of freedom interaction used as the defining contrast. For example, with two factors, A and B, the AB interaction with 4 degrees of freedom is split into defining contrasts labeled AB and AB^2, each with 2 degrees of freedom. By convention, the first letter of the interaction expression always has a power of 1 so that A^2B or A^2B^2 are not used as defining contrast expressions. The powers of A and B are the coefficients α_1 and α_2 in the defining contrast function $L = \alpha_1 x_1 + \alpha_2 x_2$.

The modulo 3 residues of $L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ are determined for 3^n factorials since there are three levels for each of the factors. The residues r(mod 3) are 0, 1 and 2. The defining contrast function for AB with $\alpha_1 = \alpha_2 = 1$ is

$$L_1 = x_1 + x_2$$

and for AB^2 with $\alpha_1 = 1$ and $\alpha_2 = 2$ the defining contrast function is

$$L_2 = x_1 + 2x_2$$

The treatments with $L_i = 0(\text{mod } 3)$ are assigned to one block, those with $L_i = 1(\text{mod } 3)$ are assigned to a second block, and those with $L_i = 2(\text{mod } 3)$ are assigned to the third block. The treatments are assigned to blocks according to the value of the residues for L_1 or L_2. The common practice is to have one-half of the replications assigned according to each of the defining contrast functions. The treatment assignments for L_1 and L_2 are shown in Display 11.4.

Display 11.4 Incomplete Block Design for a 3^2 Factorial with the AB or AB^2 Component Confounded with Blocks

AB confounded

$L_1 = x_1 + x_2$

Block 1	*Block 2*	*Block 3*
00 12 21	01 10 22	02 11 20
$L_1 = 0(\text{mod } 3)$	$L_1 = 1(\text{mod } 3)$	$L_1 = 2(\text{mod } 3)$

AB^2 confounded

$L_2 = x_1 + 2x_2$

Block 1	*Block 2*	*Block 3*
00 11 22	02 10 21	01 12 20
$L_2 = 0(\text{mod } 3)$	$L_2 = 1(\text{mod } 3)$	$L_2 = 2(\text{mod } 3)$

The residues of L_1 and L_2 for the nine treatment combinations are

x_1	x_2	L_1	r(mod 3)	L_2	r(mod 3)
0	0	0	0	0	0
0	1	1	1	2	2
0	2	2	2	4	1
1	0	1	1	1	1
1	1	2	2	3	0
1	2	3	0	5	2
2	0	2	2	2	2
2	1	3	0	4	1
2	2	4	1	6	0

Confounding with Three or More Factors

The 3^3 factorial requires 27 experimental units for a single replication. An incomplete block design with three blocks of nine experimental units each can be constructed by confounding a three-factor interaction component with blocks. The three-factor interaction with 8 degrees of freedom has four components, each with 2 degrees of freedom. For the purpose of obtaining a defining contrast the four components are designated ABC, ABC^2, AB^2C, and AB^2C^2. The defining contrast for each of these components is

$$ABC \longrightarrow L = x_1 + x_2 + x_3$$
$$ABC^2 \longrightarrow L = x_1 + x_2 + 2x_3$$
$$AB^2C \longrightarrow L = x_1 + 2x_2 + x_3$$
$$AB^2C^2 \longrightarrow L = x_1 + 2x_2 + 2x_3$$

Any component of the three-factor interaction can be used to generate three blocks of nine experimental units each. If the component AB^2C is used, the defining contrast is

$$L = x_1 + 2x_2 + x_3$$

The three blocks are constructed with treatment combinations $x_1x_2x_3$ that provide the residues for the defining contrast of $L = 0(\text{mod } 3)$, $L = 1(\text{mod } 3)$, and $L = 2(\text{mod } 3)$, respectively.

Residues are determined for each of the 27 treatment combinations—000, 001, 002, 010, ... ,222. For example, the treatment combination 000 has residue $L = 0 + 2(0) + 0 = 0(\text{mod } 3)$ and the treatment combination 021 has residue $L = 0 + 2(2) + 1 = 2(\text{mod } 3)$. Each is placed in its respective block with other treatments that have the same residues for the defining contrast. A different component of the three-factor interaction can be confounded in each replication of the experiment.

Four-factor interaction components are confounded with blocks in 3^4 factorials. The eight components each with 2 degrees of freedom are $ABCD$, AB^2CD, ABC^2D, $ABCD^2$, AB^2C^2D, AB^2CD^2, ABC^2D^2, and $AB^2C^2D^2$. Any of these components can be used to block the 81 experimental units into three blocks of 27 experimental units each.

Two Generalized Interactions Occur with Further Reductions in Block Size

The number of incomplete blocks remains a multiple of 3 with the 3^n factorials upon reduction of block sizes. A second defining contrast is confounded with blocks to reduce block sizes from nine experimental units to three experimental units for a 3^3 factorial. In the case of 3^n factorials, there are two generalized interactions for each pair of defining contrasts.

If two defining contrasts are chosen, say X and Y, then the generalized interactions are given as the symbolic products XY and XY^2. The exponents of the symbolic products are reduced modulo 3 to derive the generalized interactions. Suppose the two-factor interaction components $X = AB$ and $Y = AC^2$ are used as defining contrasts.

The XY symbolic product is

$$XY = AB \times AC^2 = A^2BC^2$$

Since the exponent of the first term of the symbolic product A is not unity, upon reduction modulo 3, the product is squared and the exponents are reduced modulo 3 as follows:

$$(A^2BC^2)^2 = A^4B^2C^4 = AB^2C$$

where the last term results from reducing the exponents of the previous term modulo 3.

The XY^2 symbolic product is

$$XY^2 = AB \times (AC^2)^2 = AB \times A^2C^4 = A^3BC^4$$

and upon reducing the exponents of the last term modulo 3, the result is

$$XY^2 = BC$$

Upon choosing the two defining contrasts, AB and AC^2, two generalized interactions, AB^2C and BC, are also confounded with the nine blocks of three experimental units each. Care must be exercised in the choice of defining contrasts to avoid confounding a main effect with blocks. For example, the choice of two three-factor interaction components, ABC and AB^2C^2, results in a generalized interaction of

$$XY = ABC \times AB^2C^2 = A^2B^3C^3 \longrightarrow (A^2B^3C^3)^2 = A^4B^6C^6$$

When the exponents of the squared symbolic product are reduced modulo 3 the generalized interaction is the main effect A.

Suppose the two interaction components AB and AC^2 are chosen to construct nine blocks of three units each for the 3^3 factorial. The defining contrasts are

$$L_1 = x_1 + x_2$$

and

$$L_2 = x_1 + 2x_3$$

There will be nine pairs of residues modulo 3 from the two defining contrasts, and three treatment combinations will be associated with each of the pairs of residues. For example, the residues $(L_1, L_2) = (2, 2)$ occur with treatment combinations (021), (112), and (200). Those three treatment combinations are placed in the same block.

11.6 Concluding Remarks

As we have seen in this chapter, the factorials are versatile treatment designs that can be adapted to a variety of experimental blocking conditions. The use of incomplete block designs results in some factorial effects confounded with blocks, either completely or partially, and increased complexity in the analysis of the results with the partially confounded designs. The goal of reducing block size is to reduce the experimental error variance. If the reduction in the estimate of experimental error variance is sufficient to overcome the loss in some information on the confounded factorial effects, then the use of an incomplete block design has been worthwhile. Cochran and Cox (1957) have tabled a number of incomplete block designs for symmetrical and asymmetrical factorials and provide some examples of the analysis with several of these designs. Detailed information on the underlying principles of design construction can be found in Kempthorne (1952), John (1987), and John and Williams (1995).

EXERCISES FOR CHAPTER 11

1. Construct one replication of an incomplete block design for a 2^4 factorial in two blocks of eight experimental units each with $ABCD$ confounded with blocks. Randomly assign the treatments to the eight experimental units in each block.

2. Suppose an incomplete block design is required for a 2^5 factorial, and the blocks cannot exceed eight experimental units in size.
   a. How many blocks will there be?
   b. How many defining contrasts are required for one replication of the experiment?
   c. How many other effects, generalized interactions, will be confounded with blocks?

d. Choose some defining contrast(s) to construct one replication of the design, and determine what other interaction(s), if any, also will be confounded with blocks.

3. Four replications of a 2^4 factorial experiment are required. The design must be conducted with incomplete blocks of eight experimental units. Construct a design such that no effect is completely confounded with blocks in the experiment.

4. A 2^7 factorial experiment is to be conducted in eight blocks of 16 experimental units, each using $ABCD$, $ABEF$, and EFG as defining contrasts. What other effects are confounded with blocks?

5. Suppose a 2^6 factorial experiment was conducted in eight blocks of eight experimental units each, and $BCDE$, $ABDE$, and ADE were used as defining contrasts.
   a. What other effects were confounded with blocks?
   b. Could there have been a better choice of defining contrasts for this design? Explain.
   c. If you have decided that a better choice is possible, then choose a set that will improve the design and defend your choice.

6. An animal scientist conducted a study on the effects of heat stress and dietary intake of protein and saline water on laboratory mice. The three factors were each used at two levels in a 2^3 factorial arrangement. The levels for the factors were (A) protein (low, high); (B) water (normal, saline); and (C) heat stress (room temperature, heat stress). An incomplete block design was used with four mice from an individual litter used in each block. Each mouse was put in an individual cage and assigned one of the treatments. One replication of the experiment consisted of two litters of mice. The weight gains (grams) for the mice are shown for each mouse next to the treatment identification.

Litter 1		*Litter 2*		*Litter 3*		*Litter 4*		*Litter 5*		*Litter 6*	
(1)	27.5	*ab*	24.3	*bc*	19.5	*abc*	19.7	(1)	24.5	*a*	33.1
bc	20.6	*c*	24.3	*a*	24.1	*b*	19.5	*c*	23.0	*b*	20.5
abc	22.0	*ac*	22.8	*ab*	22.4	(1)	22.5	*ab*	23.4	*ac*	19.8
a	28.6	*b*	24.6	*c*	22.0	*ac*	18.8	*abc*	21.7	*bc*	18.5
Replicate I				Replicate II				Replicate III			

   a. Which treatment effects are confounded with blocks (litters)?
   b. Estimate the factor effects and interactions and their standard errors, and compute the analysis of variance for these data.
   c. Interpret the results.

7. An experiment was conducted to investigate the effects of four factors on the operation of a metal lathe. The four factors each at two levels were (A) speed of lathe rotation (60, 75); (B) angle of cut (30, 45); (C) frequency of lubrication (10 sec, 30 sec); and (D) alloy for cutting tip (1 and 2). The factor levels were used in a 2^4 factorial arrangement for the experiment. Only eight cutting trials could be run in a single day. An incomplete block design with two blocks (days) in each of two replications was set up with $ABCD$ as the defining contrast. The cutting tip wear for each of

the treatments in each block follows. Days 1 and 2 constitute the first replication and Days 3 and 4 constitute the second replication.

Cutting Tip Wear

Day 1		*Day 2*		*Day 3*		*Day 4*	
(1)	40	*a*	24	(1)	43	*a*	28
ab	33	*b*	31	*ab*	30	*b*	35
ac	31	*c*	27	*ac*	30	*c*	28
bc	38	*abc*	23	*bc*	32	*abc*	20
ad	22	*d*	48	*ad*	26	*d*	44
bd	37	*abd*	35	*bd*	33	*abd*	36
cd	49	*acd*	29	*cd*	40	*acd*	25
abcd	30	*bcd*	37	*abcd*	31	*bcd*	34

   a. Compute the analysis of variance for the data.
   b. Compute the factor effects and their standard errors.
   c. Interpret the results.

8. Design an incomplete block design for a 3^3 factorial that has nine experimental units per block in four replications of three blocks each. Confound a different component of the three-factor interaction in each replication.

9. An investigator needs an incomplete block design for a 3^3 factorial that has three experimental units in each block. Consider the two pairs of defining contrasts to construct the design:
   a. ABC, AB^2C
   b. AB, AC^2

   Which is the better pair of defining contrasts to construct the design? Explain.

11A Appendix: Incomplete Block Design Plans for 2^n Factorials

Table 11A.1 Number of factors and blocks, block sizes, defining contrasts, and generalized interactions to construct incomplete block designs with 2^n factorials

Factors n	*Blocks* 2^{n-q}	*Block Size* $k = 2^q$	*Defining Contrasts*	*Generalized Interactions*
4	2	8	*ABCD*	
	4	4	*ABC, ABD*	*CD*
5	2	16	*ABCDE*	
	4	8	*ABC, CDE*	*ABDE*
	8	4	*ABC, ACD, ADE*	*BD, CE, ABE, BCDE*
6	2	32	*ABCDEF*	
	4	16	*ABCD, CDEF*	*ABEF*
	8	8	*ACE, ABEF, ABCD*	*ADF, BCF, BED, CDEF*
	16	4	*ABF, ACF, CDF, DEF*	*AD, CE, BC, BE, AEF, BDF, ABCD, ABDE, ACDE, ABCEF, BCDEF*
7	2	64	*ABCDEFG*	
	4	32	*ABCDE, ABEFG*	*CDFG*
	8	16	*ABG, CDE, EFG*	*ABEF, CDFG, ABCDF, ABCDEG*
	16	8	*ABC, ADG, CDE, DEFG*	*AEF, BDF, BEG, CFG, ABDE, ABFG, ACDF, ACEG, BCDG, BCEF, ABCDEFG*
	32	4	*ABG, BCG, CDG, DEG, EFG*	*AC, BD, BF, DG, CE, ADG, AFG, BEG, CFG, ABCD, ABEF, ABFG, ACDF, ABDE, ACEG, ADEF, BCDE, BCEF, CDEF, ABCEG, ABDFG, ACDEG, ACEFG, BCDFG, BDEFG, ABCDEFG*

12 Fractional Factorial Designs

Discussions on the versatility of 2^n factorial treatment designs are continued in this chapter. Experiments using only a fractional replication of the factorial arrangement are proposed as a means of effectively obtaining information on factors in the early stages of experimentation. Methods for constructing the designs are described and the analysis to estimate and test significance of factorial effects is illustrated for large 2^n factorial experiments without complete replication.

12.1 Reduce Experiment Size with Fractional Treatment Designs

The 2^n factorial treatment designs are useful for conducting preliminary studies with many factors to identify the more important factors and factor interactions. However, the number of experimental units increases geometrically with the number of factors in the study.

Fractional factorial designs use only one-half, one-fourth, or even smaller fractions of the 2^n treatment combinations. They are used for one or more of several reasons, including

- the number of treatments required exceeds resources
- information is required only on main effects and low-order interaction
- screening studies are needed to check on many factors
- an assumption is made that only a few effects are important

Some industrial research and development studies can exceed the capacity of the research facility if all treatment combinations are included in the experiment for