$$Z = M'(x - x_s) \tag{13A.8}$$

where the columns of M are the normalized eigenvectors m_i.

The various methods required for the calculations can be found in standard matrix algebra books such as Graybill (1983). The matrix calculations can be performed with many of the common computer programs. The calculations are illustrated with the response equation for Example 13.3,

$$\hat{y} = 169 + 6.747x_1 + 26.385x_2 - 10.875x_1^2 - 21.625x_2^2 - 15.25x_1x_2$$

The determinantal equation is

$$\begin{vmatrix} -10.875 - \lambda & -7.625 \\ -7.625 & -21.625 - \lambda \end{vmatrix} = 0$$

with $\lambda^2 + 32.5\lambda + 177.03 = 0$. The roots of the quadratic equation are $\lambda_1 = -25.58$ and $\lambda_2 = -6.92$. Thus, the canonical equation with $\hat{y}_s = 177.25$ is

$$\hat{y} = 177.25 - 25.58Z_1^2 - 6.92Z_2^2$$

The matrix of normalized eigenvectors is

$$M = \begin{bmatrix} 0.4603 & 0.8877 \\ 0.8877 & -0.4603 \end{bmatrix}$$

The coordinate of the stationary point is $x_{1s} = -0.156$ and $x_{2s} = 0.665$, and the relationship between the canonical variables and the coded factor variables is

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 0.4603 & 0.8877 \\ 0.8877 & -0.4603 \end{bmatrix} \begin{bmatrix} (x_1 + 0.156) \\ (x_2 - 0.665) \end{bmatrix}$$

or

$$Z_1 = 0.4603(x_1 + 0.156) + 0.8877(x_2 - 0.665)$$

$$Z_2 = 0.8877(x_1 + 0.156) - 0.4603(x_2 - 0.665)$$

14 Split-Plot Designs

This chapter introduces the split-plot design for experiments with a factorial treatment design and describes some unique features of the design relative to its structure, composition of experimental errors, and analysis. The relative efficiency for split-plot designs is also discussed. Extensions and variations of the design include the split-split-plot and split-block designs.

14.1 Plots of Different Size in the Same Experiment

One factor sometimes requires more experimental material for its evaluation than a second factor in factorial experiments. In agronomic or horticultural field trials a factor such as cultural methods may require the use of equipment that is best-suited for large plots, whereas another factor in the experiment such as cultivar or fertility level may be applied easily to a much smaller plot of land. The larger cultural treatment plot, the *whole plot*, is split into smaller *subplots* to which the different cultivars or fertility treatments are applied. This is known as a *split-plot* design, and in this particular example there are two different sizes of experimental units.

The experiment used for the following example illustrates the creation of a split-plot design when a second factor was introduced to subdivisions of the existing experimental units for an experiment already in progress.

Example 14.1 Nitrogen Fertilizer and Thatch Accumulation in Penncross Creeping Bent Grass

The soil for most golf greens is almost pure sand and frequent irrigation and fertilization are required to maintain the turf. The sandy soil has little capacity to retain nitrogen, and after fertilization the nitrogen quickly leaches from the root zone after irrigation. Administering large initial doses of nitrogen to retain nitrogen for a longer period of time is harmful to the turfgrass and soil microbes. Thus, a means to apply moderate rates of nitrogen and to retain nitrogen in the root zone would be very beneficial.

Nitrogen fertilizers are manufactured in various chemical configurations. Two commonly used fertilizers, ammonium sulphate and urea, were known to be fast-release nitrogen forms and were expected to leach out of the soil very quickly. Others such as isobutylidene diurea (IBDU) and sulphur-coated urea, urea(SC), although more costly, were known to be slow-release nitrogen forms and expected to stay in the soil for a longer period of time.

A second factor that could affect nitrogen retention was the turf thatch, or the buildup of dead grass. The thatch was removed frequently because it was thought plant diseases accumulated in the buildup. However, thatch provides a capacity to retain nitrogen fertilizer and could partially relieve some of the difficulty of nitrogen loss with the sandy soil.

Research Objective: A soil scientist wanted to investigate the effects of nitrogen supplied in different chemical forms and later evaluate those effects combined with those of thatch accumulation on the quality of an established turf.

Treatment Design: The four forms of nitrogen fertilizer used for the study were (1) urea, (2) ammonium sulphate, (3) isobutylidene diurea (IBDU), and (4) sulphur-coated urea, urea(SC). Each of the fertilizers was to be supplied to the turf at a rate of 1 pound of nitrogen per 1000 square feet of turf. Any differences in responses to the fertilizers then could be attributed to the mode of nitrogen release because an equivalent amount of nitrogen was supplied by each form.

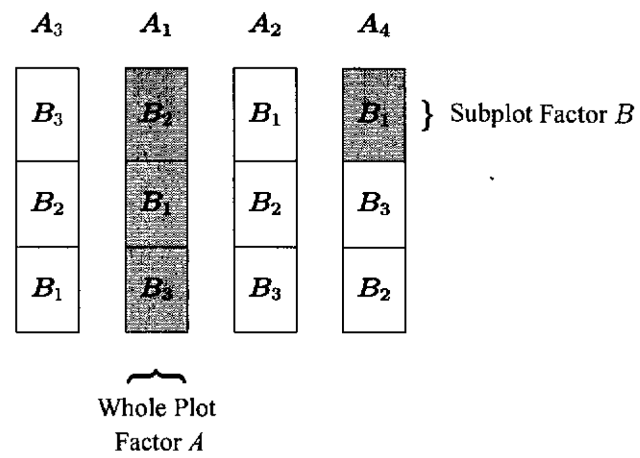
Experiment Design: A golf green had been constructed and seeded with Penncross creeping bent grass on the experimental plots. The nitrogen treatment plots were arranged on the golf green in a randomized complete block design with two replications.

After two years the second treatment factor, years of thatch accumulation, was added to the experiment. Each of the eight experimental plots was split into three subplots to which levels of the second treatment factor were randomly assigned. The lengths of time the thatch was allowed to accumulate on the subplot green were two, five, or eight years.

The resulting split-plot design had the whole-plot treatment factor of nitrogen source in a randomized complete block design with years of thatch accumulation as a subplot treatment factor.

The field plan for one replication of the split-plot design for the turfgrass experiment after randomization is shown in Display 14.1. The nitrogen whole-plot treatment, factor A, has four levels, and the years of thatch accumulation subplot factor, factor B, has three levels.

Display 14.1 Split-Plot Experimental Layout for the Turfgrass Experiment



Source: Dr. I. Pepper, Department of Soil and Water Science, University of Arizona.

An Altered Randomization for Split-Plot Designs

The usual randomization of the factorial treatment combinations to the experimental units has been altered to accommodate the particular requirements of the experiment. For example, the split-plot design in Display 14.1 has one additional restriction in the randomization relative to that of the usual randomized design. In the usual randomized design the 12 treatment combinations are assigned randomly to the 12 subplots; however, in the split-plot design the three levels of factor B combined with a single form of nitrogen, factor A, are restricted to the same whole plot. Thus, the split-plot design is a product of specific changes in the random allocation of factorial treatment combinations to the experimental units.

More Possibilities for Two Unit Sizes

If one factor, such as temperature, humidity, or photoperiod, requires environmental control chambers, then each level of that factor requires a separate chamber. A second treatment factor such as culture media for plants and even a third treatment factor can be included within each of the chambers, thereby achieving an economical use of the chambers to study more than one factor at a time. The chambers represent whole plots, and the subplots are units to which the second factor levels are applied inside each chamber.

In education research a teaching method is applied as a whole-plot treatment to an entire classroom; however, subgroups of students within each classroom can be used as subplots to study an additional factor, such as the use of certain library materials or microcomputers.

Industrial research may require replicate batches of raw material mixtures. The whole-plot treatment, such as product mixtures, can be one batch of raw material, and a subplot treatment, such as curing time, can be applied to subbatches of the larger product mixture batches.

4.2 Two Experimental Errors for Two Plot Sizes

The statistical analysis must take into account the presence of two different types or sizes of experimental units in the experiment. Factor A effects are estimated from the whole plots for factor A. The factor B effects and the AB interaction effects are estimated from the subplots for factor B. Since the whole plots and subplots are experimental units of different sizes or types, they have different precision. The differing precision must be taken into account for making comparisons among treatment means.

The consideration of two separate errors is a consequence of the fact that observations from different subplots in the same whole plot may be positively correlated. The correlation reflects the nature of experimental units to respond similarly when they are adjacent to one another, such as neighboring field plots, students in a classroom, cultures in a growth chamber, or units from the same batch of raw materials in the industrial experiment.

We assume a correlation ρ between observations on any two subplots in the same whole plot and also assume observations from two different whole plots are uncorrelated. Given these assumptions, it can then be shown that the error variance for the main effects of A on a per-subplot basis is $\sigma^2[1+(b-1)\rho]$ if there are b subplots in each whole plot. Likewise, for comparisons among the main effects of B and AB interaction comparisons the error variance per subplot is $\sigma^2(1-\rho)$.

As a consequence of these differences in the errors associated with the whole plot and subplot treatment comparisons, the partition of the sum of squares in the analysis of variance is altered somewhat from the usual partition for a two-factor factorial design. The partitions for factor effects and blocking factors remain the same as that for the usual factorial designs, but the experimental error is partitioned into two components. One component of experimental error is associated with the whole-plot treatment factor, and the other component is associated with the subplot treatment factor and interaction.

14.3 The Analysis for Split-Plot Designs

The Split-Plot Model

A mixed-model formulation is used for the split-plot design to reflect the separate experimental error variances for the subplot and whole-plot units. It includes separate random error effects for the whole-plot units and the subplot units. If the whole-plot treatment factor is placed in a randomized complete block design the linear model is

$$y_{ijk} = \mu + \alpha_i + \rho_k + d_{ik} + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad \textbf{(14.1)}$$
$$i = 1,2,\ldots,a \qquad j = 1,2,\ldots,b \qquad k = 1,2,\ldots,r$$

where μ is the general mean, α_i is the effect of the ith level of factor A, ρ_k is the effect of the kth block, d_{ik} is the whole-plot random error, β_j is the effect of the jth level of factor B, $(\alpha\beta)_{ij}$ is the interaction effect between factors A and B, and e_{ijk} is the subplot random error.

The whole-plot and subplot errors are assumed to be independent, normally distributed random errors with mean 0 and variances σ_d^2 and σ_e^2, respectively. Randomization of treatments to the experimental units justifies the assumption of independence for the two random errors and the equal correlation between the errors for subplot units on the same whole-plot unit.

The Split-Plot Analysis of Variance

The expectation of the mean squares for the analysis of variance using the mixed model in Equation (14.1) is shown in Table 14.1 with fixed effects for factors A and B.

Table 14.1 Expected mean squares for the split-plot analysis of variance

Source of Variation	*Degrees of Freedom*	*Mean Square*	*Expected Mean Square*
Blocks	$r-1$	*MS* Blocks	
A	$a-1$	*MSA*	$\sigma_e^2 + b\sigma_d^2 + rb\theta_a^2$
Error(1)	$(a-1)(r-1)$	*MSE*(1)	$\sigma_e^2 + b\sigma_d^2$
B	$b-1$	*MSB*	$\sigma_e^2 + ra\theta_b^2$
AB	$(a-1)(b-1)$	*MS*(**AB**)	$\sigma_e^2 + r\theta_{ab}^2$
Error(2)	$a(r-1)(b-1)$	*MSE*(2)	σ_e^2

The expected mean squares for Error(1) and Error(2) based on the mixed model reflect the differences in the variability for the two different types of experimental units. The expected error variances for the whole plots are larger than those for the subplots.

Tests of Hypotheses About Factor Effects

The F statistics to test the null hypotheses for interaction and main effects are

- (interaction) H_0: $(\alpha\beta)_{ij} = 0$ versus H_a: $(\alpha\beta)_{ij} \neq 0$ for some i, j

$$F_0 = \frac{MS(\boldsymbol{AB})}{MSE(2)} \text{ with } (a-1)(b-1) \text{ and } a(r-1)(b-1) \text{ d.f.}$$

- (factor B main effects) H_0: $\overline{\mu}_{.1} = \cdots = \overline{\mu}_{.b}$ versus H_a: $\overline{\mu}_{.i} \neq \overline{\mu}_{.j}$ for some i, j

$$F_0 = \frac{MS\boldsymbol{B}}{MSE(2)} \text{ with } b-1 \text{ and } a(r-1)(b-1) \text{ d.f.}$$

- (factor A main effects) H_0: $\overline{\mu}_{1.} = \cdots = \overline{\mu}_{a.}$ versus H_a: $\overline{\mu}_{i.} \neq \overline{\mu}_{j.}$ for some i,j

$$F_0 = \frac{MS\boldsymbol{A}}{MSE(1)} \text{ with } a-1 \text{ and } (a-1)(r-1) \text{ d.f.}$$

Example 14.2 Data and Analysis

One of the measurements made on the turfgrass plots of Example 14.1 was the chlorophyll content of clippings (mg/g) sampled from each plot. The data are shown in Table 14.2, and the analysis of variance is shown in Table 14.3.

Table 14.2 Chlorophyll content (mg/gm) of grass clippings

		Years of Thatch Accumulation		
Nitrogen Source	*Block*	2	5	8
Urea	1	3.8	5.3	5.9
	2	3.9	5.4	4.3
Ammonium sulphate	1	5.2	5.6	5.4
	2	6.0	6.1	6.2
IBDU	1	6.0	5.6	7.8
	2	7.0	6.4	7.8
Urea (SC)	1	6.8	8.6	8.5
	2	7.9	8.6	8.4

Table 14.3 Analysis of variance of chlorophyll content of Penncross creeping bent grass clippings

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	23	48.78			
Block	1	0.51	0.51		
Nitrogen (*N*)	3	37.32	12.44	29.62	0.010
Error (1)	3	1.26	0.42		
Thatch (*T*)	2	3.82	1.91	9.10	0.009
$N \times T$	6	4.15	0.69	3.29	0.065
Error (2)	8	1.72	0.21		

Computational Notes

The analysis of variance for the split-plot design can be computed with many available statistical computing packages. The instructions for the analysis are equivalent to those for a two-factor factorial treatment design except for the necessity to compute two separate error terms for the analysis.

The sum of squares for Error(1) is numerically equivalent to the error sum of squares for the experiment design that was utilized for the whole plots. With completely randomized designs it is equivalent to the sum of squares for whole plots within factor A treatments. For randomized complete block designs it is equivalent to the computation of the sum of squares for Blocks × A interaction. The sum of squares for Error(2) is ordinarily the residual sum of squares computed automatically by the program.

Alternatively, the analysis can be conducted with statistical programs specifically written for mixed models, which have been incorporated into many statistical computing packages. They can be particularly useful for split-plot designs in blocked designs with random blocks. The statistical estimation in these programs is based on maximum likelihood methods, which are beyond the scope of this book.

The test for interaction between nitrogen and thatch is $F_0 = 0.69/0.21 = 3.29$, and the test is not significant with $Pr > F = .065$ (Table 14.3). The test for differences among the thatch means is $F_0 = 1.91/0.21 = 9.10$, and it is significant with $Pr > F = .009$. The test for differences among the whole-plot nitrogen treatment means is $F_0 = 12.44/0.42 = 29.62$, and it is significant with $Pr > F = .01$.

The cell means and the marginal means for source of nitrogen and years of thatch accumulation are shown in Table 14.4. Turf that received the sulphur-coated urea, urea(SC), had the highest chlorophyll content followed by IBDU and ammonium sulphate while urea resulted in the lowest amount of chlorophyll. This relative ranking of the chlorophyll content by source of nitrogen was the same for each year of thatch accumulation.

The graph in Figure 14.1 shows a plot of the mean chlorophyll content versus years of thatch accumulation separately for each of the nitrogen sources. The marginal means in Table 14.4 indicate an increase in chlorophyll content as the

Table 14.4 Mean chlorophyll content (mg/g) of Penncross creeping bent grass clippings

	Years of Thatch Accumulation			
Nitrogen Source	*2*	*5*	*8*	*Nitrogen Means*
Urea	3.85	5.35	5.10	4.77
Ammonium sulphate	5.60	5.85	5.80	5.75
IBDU	6.50	6.00	7.80	6.77
Urea(SC)	7.35	8.60	8.45	8.13
Thatch means	5.83	6.45	6.79	

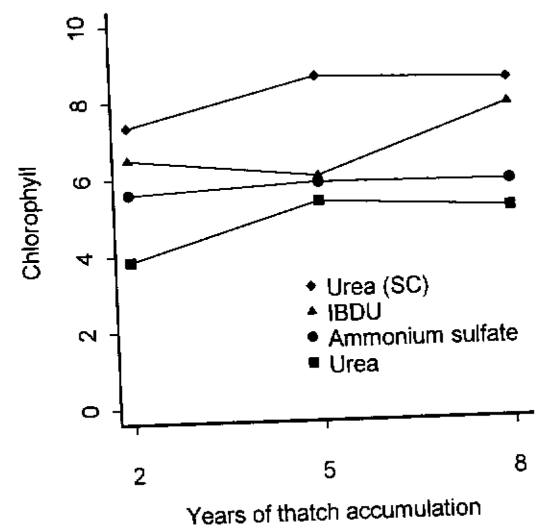


Figure 14.1 Mean chlorophyll content of Penncross creeping bent grass clippings versus nitrogen source for two, five, and eight years of thatch accumulation

years of thatch accumulation increase. However, observation of the cell means and their plotted values in Figure 14.1 shows that this general trend was not manifested for each of the separate sources of nitrogen.

This difference in the trends among the nitrogen sources indicates the possibility that some form of interaction may be present even though the global test for interaction was not significant. Partitions of the interaction sum of squares will sometimes reveal the presence of certain forms of interaction between the factors.

Interpretation with Regression Contrasts

The factorial treatment design is composed of one quantitative factor, years of thatch accumulation (T), and one qualitative factor, source of nitrogen (N). The treatment design lends itself to the use of the polynomial regression partitions in the analysis of variance for one quantitative factor with one qualitative factor (discussed in Chapter 6). Linear and quadratic regression sums of squares partitions can be computed for years of thatch accumulation (T) with corresponding partitions for the nitrogen × thatch interaction. These partitions are shown in the analysis of variance for the experiment in Table 14.5.

Table 14.5 Analysis of variance for chlorophyll content of Penncross creeping bent grass clippings with orthogonal polynomial regression partitions for thatch factor

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	23	48.78			
Blocks	1	0.51	0.51		
Nitrogen (N)	3	37.32	12.44	29.62	0.010
Error(1)	3	1.26	0.42		
Thatch (T)	2	3.82	1.91	9.10	0.009
T linear	1	3.71	3.71	17.67	0.003
T quadratic	1	0.11	0.11	0.52	0.494
$N \times T$	6	4.15	0.69	3.29	0.065
$N \times T$ linear	3	0.80	0.27	1.29	0.358
$N \times T$ quadratic	3	3.36	1.12	5.33	0.028
Error(2)	8	1.72	0.21		

The interaction for quadratic deviations of nitrogen × thatch is significant with $F_0 = 1.12/0.21 = 5.33$ and $Pr > F = .028$. The linear regression partition for thatch (T) is significant with $F_0 = 3.71/0.21 = 17.67$ and $Pr > F = .003$. Thus, there are significant quadratic deviations from the linear response to years of thatch accumulation that differ among the sources of nitrogen. The differing patterns were observed in Figure 14.1.

Several types of comparisons may be of interest at this point to further clarify the interpretation. Given there was a significant regression interaction component, it will be beneficial to make a comparison among the nitrogen means for each of the years of thatch accumulation to determine if the sulphur-coated urea always resulted in the highest chlorophyll content of the turf. Also a quadratic regression of chlorophyll content on years of thatch accumulation can be computed for each of the nitrogen sources to characterize the differences among the nitrogen sources relative to years of thatch accumulation.

14.4 Standard Errors for Treatment Factor Means

The standard errors shown in Table 14.6 are used for tests of hypotheses for comparisons among the estimated treatment means. The degrees of freedom associated with each of the standard errors are those for the mean square used in the standard error. The only exception is the final comparison shown in the table where the standard error is a weighted combination of the two mean squares for error. Consequently, the appropriate degrees of freedom can be approximated by the procedure introduced by Satterthwaite (1946) (Chapter 5). The approximation is

$$\text{d.f.} = \frac{[(b-1)MSE(2) + MSE(1)]^2}{\dfrac{[(b-1)MSE(2)]^2}{f_2} + \dfrac{[MSE(1)]^2}{f_1}} \tag{14.2}$$

where f_1 and f_2 are the degrees for $MSE(1)$ and $MSE(2)$, respectively, from the analysis of variance.

Table 14.6 Standard error estimators for the split-plot design ($\boldsymbol{A}$ = whole-plot factor, $\boldsymbol{B}$ = subplot factor)

Treatment Comparison	*Standard Error Estimator*
Difference between two A means $\bar{y}_{u..} - \bar{y}_{v..}$	$\sqrt{\dfrac{2MSE(1)}{rb}}$
Difference between two B means $\bar{y}_{.u.} - \bar{y}_{.v.}$	$\sqrt{\dfrac{2MSE(2)}{ra}}$
Difference between two B means at the same level of A $\bar{y}_{ju.} - \bar{y}_{ju.}$	$\sqrt{\dfrac{2MSE(2)}{r}}$
Difference between two A means at the same or different level of B $\bar{y}_{uk.} - \bar{y}_{vk.}$ or $\bar{y}_{uk.} - \bar{y}_{vm.}$	$\sqrt{\dfrac{2[(b-1)MSE(2) + MSE(1)]}{rb}}$

It should be noted that comparisons requiring standard errors based on the linear combination of mean squares, $(b-1)MSE(2) + MSE(1)$, only have approximate probability levels for tests of hypotheses and confidence intervals. The linear

combination of mean squares no longer shares the same probability distribution properties of the individual mean squares.

Standard Errors for the Turfgrass Experiment

Referring to Table 14.6 the standard errors required for the analysis of Example 14.1 are differences between

- two nitrogen means

$$\sqrt{\frac{2MSE(1)}{rb}} = \sqrt{\frac{2(0.42)}{6}} = 0.37 \text{ with 3 degrees of freedom}$$

- two thatch means

$$\sqrt{\frac{2MSE(2)}{ra}} = \sqrt{\frac{2(0.21)}{8}} = 0.23 \text{ with 8 degrees of freedom}$$

- two thatch means for the same nitrogen

$$\sqrt{\frac{2MSE(2)}{r}} = \sqrt{\frac{2(0.21)}{2}} = 0.46 \text{ with 8 degrees of freedom}$$

- two nitrogen means at the same or different thatch levels

$$\sqrt{\frac{2[(b-1)MSE(2) + MSE(1)]}{rb}} = \sqrt{\frac{2[2(0.21) + 0.42]}{6}} = 0.53$$

where the degrees of freedom for the last standard error obtained from the Satterthwaite approximation are

$$\text{d.f.} = \frac{[2(0.21) + 0.42]^2}{\dfrac{[2(0.21)]^2}{8} + \dfrac{[0.42]^2}{3}} = 8.73 \text{ or } 9 \tag{14.3}$$

The experimental error variance for subplots, $MSE(2) = 0.21$, is one-half the magnitude of experimental error variance estimated for the whole plots, $MSE(1) = 0.42$. Thus, the comparisons among years of thatch accumulation means and interaction means will be more precise than comparisons among source of nitrogen means. The question of whether there has been an equitable trade-off between the advantages and disadvantages of splitting the plots will be partially answered by the relative efficiency computations in Section 14.6.

14.5 Features of the Split-Plot Design

Randomization for the split-plot design requires that levels of factor A be randomized to the whole units in accordance with the protocol for the experiment design in which the whole units are arranged—that is, completely randomized, randomized complete block, and so forth. Levels of factor B are assigned to the subunits *within* each of the whole units at random, separately for each of the whole units.

The design also can be described as a confounded factorial design (discussed in Chapter 11). The subunits are regarded as the experimental units, so that the levels of factor A are applied to groups or blocks of the subunits. Consequently, comparisons among the levels of A are confounded with the blocks of subunits. The split-plot design is often referred to as a *confounded factorial design* in which the *main effects* are confounded with blocks, whereas confounding was restricted to *interactions* for the designs in Chapter 11.

The Analysis of Variance for Common Experiment Designs

The sources of variation and degrees of freedom for the sum of squares partitions are shown in Table 14.7 for the split-plot design in which the whole plots are arranged in three of the most common experiment designs—the completely randomized, randomized complete block, and Latin square designs. Each design has r replications, a levels of A, and b levels of B.

Table 14.7 Sources of variation and degrees of freedom for the split-plot design analysis of variance

Completely Randomized		*Randomized Complete Block*		*Latin Square*	
		Whole Plots			
Source	*d.f.*	*Source*	*d.f.*	*Source*	*d.f.*
				Rows	$a-1$
		Blocks	$r-1$	Columns	$a-1$
A	$a-1$	A	$a-1$	A	$a-1$
Error(1)	$a(r-1)$	Error(1)	$(r-1)(a-1)$	Error(1)	$(a-1)(a-2)$
		Subplots			
B	$b-1$	B	$b-1$	B	$b-1$
AB	$(a-1)(b-1)$	AB	$(a-1)(b-1)$	AB	$(a-1)(b-1)$
Error(2)	$a(r-1)(b-1)$	Error(2)	$a(r-1)(b-1)$	Error(2)	$a(a-1)(b-1)$

Different Precision for Whole-Plot and Subplot Factor Means

The larger degrees of freedom associated with the estimates of experimental error for the factor B means and the AB interaction means and comparisons among them indicate that the B and AB effects are more precisely estimated than the A effects measured on the whole plots. However, this can be misleading as both $MSE(1)$ and $MSE(2)$ for the split-plot design have fewer degrees of freedom than does the mean square for experimental error in the design that would occur without the split-plot feature for the two-factorial effects. For example, the same factorial design in a randomized complete block design without the restrictions of the split-plot design would have $(r-1)(ab-1)$ degrees of freedom for experimental error, which exceeds that for either $MSE(1)$ or $MSE(2)$ with $(r-1)(a-1)$ and $a(r-1)(b-1)$ degrees of freedom, respectively, for the split-plot design.

Practical experience with split-plot designs has shown that $MSE(2)$ often is smaller than $MSE(1)$ as anticipated from the expected mean squares. Consequently, there can be an increase in the precision of estimates of B and AB effects that offsets the loss in degrees of freedom. However, the average experimental error over all treatment effects is the same with or without the split-plot feature. Thus, there is no net gain from the split-plot design. Rather, if there is a gain in the precision on B and AB effects it is offset by a loss in the precision on A effects. Consequently, large and interesting A effects possibly may be judged nonsignificant. Evaluation of the relative efficiency of the split-plot design in this context is discussed in Section 14.6.

Advantages of the Split-Plot Experiment

The primary advantages of the split-plot have already been mentioned: when one factor requires considerably more experimental material than another factor, such as in the agronomic field studies, or when there is an opportunity to study responses to a second factor while efficiently utilizing resources, such as in the growth chamber studies. The experiment in Example 14.1 illustrated the creation of a split-plot design by introducing a second factor to subdivisions of the existing experimental units of an experiment already in progress.

Disadvantages of the Split-Plot Experiment

The primary disadvantages of the split-plot design include the potential loss in precision in treatment comparisons and an increase in complexity of the statistical analysis. The analysis of variance and the estimation of the standard errors for different types of treatment comparisons require additional computations.

14.6 Relative Efficiency of Subplot and Whole-Plot Comparisons

Ordinarily, the relative efficiency of an experiment design refers to the efficiency as a result of blocking by some factor relative to that ignoring the blocking factor. With split-plot designs it is informative to consider the relative efficiency of using the split-plot design in lieu of the same experiment design without the split-plot feature. For example, when the whole-plot treatment factor is arranged in a randomized complete block design it is possible to determine which of the designs is more efficient for the whole-plot treatment comparisons and which is more efficient for the subplot treatments and interaction comparisons. As discussed in Section 14.5 there is a compromise between the increase in precision on the subplot treatment means and the decrease in precision on the whole-plot treatment means.

Efficiency of Subplot Comparisons

Federer (1955) shows the efficiency of the split-plot design relative to the randomized complete block design for the subplot comparisons to be

$$RE = K\frac{a(b-1)MSE(2)+(a-1)MSE(1)}{(ab-1)MSE(2)} \tag{14.4}$$

where

$$K = \frac{(f_1+1)(f_2+3)}{(f_1+3)(f_2+1)}$$

$f_1 = a(b-1)(r-1)$, the degrees of freedom for the subplot error $MSE(2)$, and $f_2 = (ab-1)(r-1)$, the degrees of freedom for the randomized complete block experimental error.

Efficiency of Whole-Plot Comparisons

The relative efficiency of the split-plot design to the randomized complete block design for the whole-plot comparisons is

$$RE = K\frac{a(b-1)MSE(2)+(a-1)MSE(1)}{(ab-1)MSE(1)} \tag{14.5}$$

with $f_1 = (a-1)(r-1)$ for the whole-plot error $MSE(1)$.

The efficiency of the split-plot design for thatch and nitrogen × thatch subplot comparisons relative to that if all 12 different combinations had been randomly allocated to the 12 plots in each block with $a = 4$, $b = 3$, $r = 2$, $f_1 = 8$, and $f_2 = 11$ is

$$RE = \frac{(9)(14)}{(11)(12)} \times \frac{8(0.21)+3(0.42)}{11(0.21)} = 0.95(1.27) = 1.21$$

or a gain of 21%. For nitrogen whole-plot comparisons with $f_1 = 3$ and $f_2 = 11$ the relative efficiency is

$$RE = \frac{(4)(14)}{(6)(12)} \times \frac{8(0.21)+3(0.42)}{11(0.42)} = 0.78(0.64) = 0.49$$

or only 49%. Therefore, a gain of only 21% in efficiency was realized for the subplot comparisons, and a loss of 51% was realized for the whole-plot comparisons with the split-plot design relative to the standard randomized complete block design.

14.7 The Split-Split-Plot Design for Three Treatment Factors

The convenience of introducing a third factor into the treatment design requires a subdivision of the subplots such that all levels of the third factor are administered to these new subdivisions, referred to as sub-subplots. The design, commonly referred to as the **split-split-plot**, has three different sizes or types of experimental units. The analysis requires the computation of an additional sum of squares for experimental error associated with the sub-subplots. The partition of the sum of squares for factors A and B is that shown in Table 14.7. If the third factor, C, has c levels the additional sources of variation and associated degrees of freedom in the analysis of variance are

Source	*Degrees of Freedom*
C	$c-1$
AC	$(a-1)(c-1)$
BC	$(b-1)(c-1)$
ABC	$(a-1)(b-1)(c-1)$
Error(3)	$ab(r-1)(c-1)$

The sum of squares for Error(3) is the usual residual sum of squares produced by a computer program. Special instructions must be given for separate computation of the sum of squares for Error(2). For randomized complete block designs the sum of squares for Error(2) is computed as the pooled sum of squares for the Blocks × B and Blocks × AB interaction.

All of the standard errors applied to A and B effects shown in Table 14.6 are valid and need only have the value of c included as part of the divisor. For example, the standard error for the difference between two factor A means $\overline{y}_{u...} - \overline{y}_{v...}$ is $\sqrt{2MSE(1)/rbc}$. The standard errors required for comparisons involving the factor C effects are shown in Table 14.8.

14.8 The Split-Block Design

Numerous variations of the basic split-plot design have been used by investigators beyond further subdivisions of the experimental units. One common variation is the split-block design in which the subunit treatments occur in a strip across the whole-plot units. The design is sometimes called a *strip plot* design.

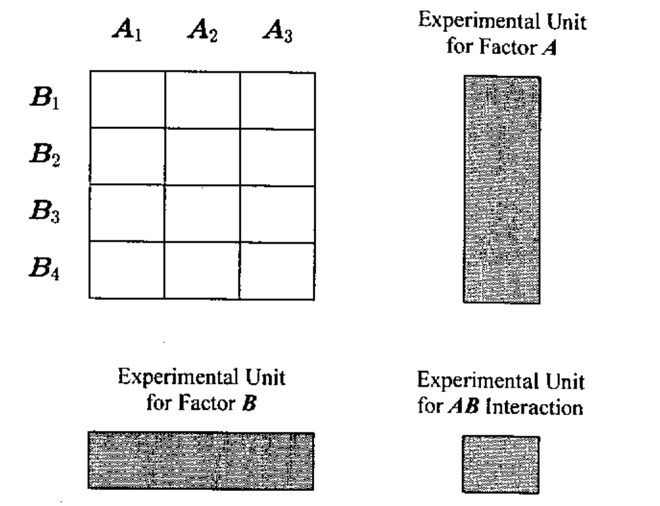
The split-block design can be useful in agricultural field studies when two treatment factors require the use of large field plots. The levels of one treatment factor A are randomly assigned to the plots in a randomized complete block design.

Table 14.8 Standard error estimators for the split-split-plot design

Treatment Comparison	*Standard Error Estimator*
Two C means ($\overline{y}_{..u.} - \overline{y}_{..v.}$)	$\sqrt{\frac{2MSE(3)}{rab}}$
Two C means at the same level of A ($\overline{y}_{i.u.} - \overline{y}_{i.v.}$)	$\sqrt{\frac{2MSE(3)}{rb}}$
Two C means at the same level of B ($\overline{y}_{.iu.} - \overline{y}_{.iv.}$)	$\sqrt{\frac{2MSE(3)}{ra}}$
Two C means at the same level of A and B ($\overline{y}_{iju.} - \overline{y}_{ijv.}$)	$\sqrt{\frac{2MSE(3)}{r}}$
Two B means at the same or different levels of C ($\overline{y}_{.ui.} - \overline{y}_{.vi.}$ or $\overline{y}_{.ui.} - \overline{y}_{.vj.}$)	$\sqrt{\frac{2[(c-1)MSE(3)+MSE(2)]}{rac}}$
Two B means at the same level of A and C ($\overline{y}_{iuj.} - \overline{y}_{ivj.}$)	$\sqrt{\frac{2[(c-1)MSE(3)+MSE(2)]}{rc}}$
Two A means at the same or different level of C ($\overline{y}_{u.i.} - \overline{y}_{v.i.}$ or $\overline{y}_{u.i.} - \overline{y}_{v.j.}$)	$\sqrt{\frac{2[(c-1)MSE(3)+MSE(1)]}{rbc}}$
Two A means at the same or different levels of B and C (for example, $\overline{y}_{uij.} - \overline{y}_{vij.}$ or $\overline{y}_{uij.} - \overline{y}_{vlk.}$)	$\sqrt{\frac{2[b(c-1)MSE(3)+(b-1)MSE(2)+MSE(1)]}{rbc}}$

The plots for the second factor B are constructed in the same manner but are laid out perpendicular to the plots for factor A. The levels of the second factor B are then randomly assigned to this second array of plots across the same block. A sketch of one block for the split-block design is shown in Display 14.2, in which there are $a = 3$ levels of factor A and $b = 4$ levels of factor B.

Display 14.2 Experimental Units for One Block of a Split-Block Design



Three Sizes of Units and Three Experimental Errors

The split-block design has three sizes of experimental units where the units for the main effects of A and B are equivalent to whole plots, each of different orientation. The unit for the AB interaction effect is a subplot where there is an intersection of the two whole plots for the respective levels of A and B. Consequently, a separate experimental error must be estimated from the analysis of variance for each of the three treatment effects.

The linear statistical model for the split-block design differs from that for the ordinary split-plot design in that it has separate random error terms for each of the three treatment effects. A blocking effect ρ_k is also included in the model

$$y_{ijk} = \mu + \rho_k + \alpha_i + d_{ik} + \beta_j + g_{jk} + (\alpha\beta)_{ij} + e_{ijk} \qquad \textbf{(14.6)}$$

$$i = 1, 2, \ldots, a \quad j = 1, 2, \ldots, b \quad k = 1, 2, \ldots, r$$

The random error effects, d_{ik}, g_{jk}, and e_{ijk} are the experimental errors for the units associated with A, B, and AB effects, respectively, with variances σ_d^2, σ_g^2, and σ_e^2. The analysis of variance for this model along with expected mean squares for fixed treatment effects is shown in Table 14.9.

Table 14.9 Analysis of variance outline for the split-block design with two treatment factors

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Expected Mean Square*
Blocks	$r-1$	SS Blocks	
A	$a-1$	SSA	$\sigma_e^2 + b\sigma_d^2 + rb\theta_a^2$
Error(1)	$(r-1)(a-1)$	$SSE(1)$	$\sigma_e^2 + b\sigma_d^2$
B	$b-1$	SSB	$\sigma_e^2 + a\sigma_g^2 + ra\theta_b^2$
Error(2)	$(r-1)(b-1)$	$SSE(2)$	$\sigma_e^2 + a\sigma_g^2$
AB	$(a-1)(b-1)$	$SS(AB)$	$\sigma_e^2 + r\theta_{ab}^2$
Error(3)	$(r-1)(a-1)(b-1)$	$SSE(3)$	σ_e^2

Standard Errors for Treatment Factor Means

The standard errors for the split-block are more complex than those for the split-plot design because of the altered randomization pattern of factor B treatment levels. The basic standard errors of the treatment means and treatment differences are shown in Table 14.10.

The degrees of freedom for the standard errors with a single mean square will be those associated with the mean square. For those standard errors with two or more mean squares it is necessary to obtain degrees of freedom values with the Satterthwaite approximation.

14.9 Additional Information About Split-Plot Designs

The basic split-plot design and several extensions of the design have been discussed in this chapter. Numerous other modifications and applications of the design have been presented elsewhere including subunit treatments in a Latin square design, confounding of comparisons among subunit treatments, and systematic arrangement of whole-unit treatments (Cochran & Cox, 1957; Petersen, 1985). Robinson (1967) gave a split-plot design and analysis for subplots in a balanced incomplete block arrangement and Coons et al. (1989) utilized a split-plot design with whole plots in a balanced incomplete block design. The statistical properties of split-plot designs with incomplete blocks were given by Mejza and Mejza (1996), and properties for the split-block designs with incomplete blocks were shown by Hering and Mejza (1997). Federer (1975) included an extensive array of modifications along with the connection to series of experiments repeated over locations or time. Little and Rubin (1987) and Jarrett (1978) provided detailed relevant discussions on the analysis of split-plot designs with missing data, including formulations to obtain standard errors for general treatment contrasts. Steel and Torrie (1980) included discussions on the analysis of repeated harvest data for perennial crops as split-plot designs. Since repeated harvests of the same plot fall under the general category of repeated measurements and longitudinal studies, a discussion of that topic is found in the next chapter.

Table 14.10 Standard error estimators for the split-block design

Treatment Comparison	*Standard Error Estimator*
Two A means $(\overline{y}_{u..} - \overline{y}_{v..})$	$\sqrt{\dfrac{2MSE(1)}{rb}}$
Two B means $(\overline{y}_{.u.} - \overline{y}_{.v.})$	$\sqrt{\dfrac{2MSE(2)}{ra}}$
Two A means, same level of B $(\overline{y}_{uj.} - \overline{y}_{vj.})$	$\sqrt{\dfrac{2[(b-1)MSE(3) + MSE(1)]}{rb}}$
Two B means, same level of A $(\overline{y}_{ju.} - \overline{y}_{jv.})$	$\sqrt{\dfrac{2[(a-1)MSE(3) + MSE(2)]}{ra}}$
Any two means, different levels of A and B $(\overline{y}_{uj.} - \overline{y}_{vk.})$	$\sqrt{\dfrac{2[aMSE(1) + bMSE(2) + (ab-a-b)MSE(3)]}{rab}}$

EXERCISES FOR CHAPTER 14

1. A split-plot experiment was conducted on sorghum with two treatment factors, plant population density and hybrid. The whole plots were used for the four levels of plant population density—10, 15, 25, and 40 plants per meter of row. There were three hybrids randomly allocated to the subplots of each plot. The experiment was conducted in a randomized complete block design with four replications. The data that follow are the weights of the seed per plant in grams.

a. Write the linear model for this experiment, explain the terms, and compute the analysis of variance for the data.

b. Construct a summary table of cell means and marginal means for this experiment, and compute the estimated standard errors for the table of means.

Head Seed Weight (g) of a Sorghum Trial					
		Plants per Meter of Row			
Hybrid	*Block*	*10*	*15*	*25*	*40*
TAM 680	1	40.7	24.2	16.1	11.2
	2	37.8	44.4	17.6	12.7
	3	32.9	27.8	19.9	14.5
	4	43.1	34.1	20.1	15.4
RS 671	1	39.4	31.3	17.9	14.8
	2	47.8	34.5	30.5	17.3
	3	44.4	25.6	22.5	17.7
	4	49.0	50.4	25.2	18.7
Tx 399	1	68.7	26.2	20.5	18.9
× Tx 2536	2	56.2	48.1	28.2	26.2
	3	44.8	41.1	30.0	19.2
	4	59.3	46.0	24.7	22.0

Source: Dr. R. Voigt, Department of Plant Sciences, University of Arizona.

c. Compute the estimated standard errors for the differences between two observed means:
   (i) for hybrids
   (ii) for plant populations
   (iii) for two hybrids at the same plant population
d. Test the hypotheses for interaction and main effects assuming fixed effects for hybrids and plant populations.
e. Compute the relative efficiency of this split-plot design for subplot and whole-plot treatments relative to the ordinary randomized complete block design, and interpret it.
f. Partition the sum of squares for plant population and the interaction sum of squares into the appropriate polynomial regression partitions, and interpret the results. Make a graph of the observed means to assist in the interpretation. The coefficients for linear, quadratic, and cubic partitions for the four levels of Plants per Meter are shown below.

Plants	10	15	25	40
Linear	− 0.546	− 0.327	0.109	0.764
Quadratic	0.513	− 0.171	− 0.741	0.399
Cubic	− 0.435	0.783	− 0.435	0.087

2. A split-plot experiment was conducted in a completely randomized design with whole-plot treatments as a 2×2 factorial (factors A and B) and the subplot treatments as three levels of factor C. There were four replications of the experimental units. Assume all treatment effects were fixed.
   a. Write the linear model for the experiment. Identify each of the model components, and show the numerical ranges on the subscript.

   b. Outline the analysis of variance table showing sources of variation, degrees of freedom, and expected mean squares.

3. Suppose the whole-plot treatments in Exercise 14.2 were arranged in a Latin square design. Repeat parts (a) and (b) in that case.

4. Suppose the subplot treatments in Exercise 14.2 were a 3×3 factorial arrangement of factors C and D while all other conditions given were the same. Repeat parts (a) and (b) in that case.

5. An investigator in food science wants to conduct an experiment to assess the effect of cold storage conditions on food quality. The two treatment factors to be used are storage temperature and container material. The food product will be placed in one of the containers and stored in a temperature control chamber for a fixed period of time after which a number of physical and subjective quality measures will be taken on the food product in each container. The investigator has three small temperature control chambers available for the experiment. The storage temperatures for the experiment are 2°C, 4°C, and 8°C. There are four container types for the study—sealed plastic, sealed cardboard covered with wax, sealed cardboard, and an open container as a control. There are four positions on the center shelf of the chamber in which the containers can be placed. Draw a diagram of a plan for the experiment so that the investigator may have three replications of the experiment. Use the following guidelines to construct your diagram.
   a. Use the labels I, II, and III to identify the three temperature chambers and the labels a, b, c, and d to identify the four positions within each chamber.
   b. Show the container type (C1, C2, C3, or C4) assigned to each position in each chamber as well as the temperature (2°, 4°, or 8°) assigned to the chamber for all three replications in your diagram.
   c. Show the randomization scheme you used to make the assignments in part (b).

6. A research specialist for a large seafood company investigated bacterial growth on oysters and mussels subjected to three different storage temperatures. Nine cold storage units were available. Three storage units were randomly selected to be used for each of the storage temperatures. Oysters and mussels were stored for two weeks in each of the cold storage units. A bacterial count was made from a sample of oysters and mussels at the end of two weeks. The logarithm of bacterial count for each sample is shown in the table at the end of the exercise.
   a. The investigator could have had three replications for the study by simply taking three random samples of each seafood from a single cold storage unit set at one temperature. In this way only three cold storage units would have been needed for the study, one for each temperature. Explain the potential difficulty with the study if it had been conducted in this way.
   b. Is there a significant increase in bacterial growth as temperature increases? Justify your answer.
   c. Is there a significant interaction between type of seafood and increase (if any) of bacterial growth with temperature? Justify your answer.
   d. Write the linear model for your analysis, state the assumptions, and explain the terms.
   e. Determine whether your assumptions about the linear model are correct for this data.

Storage Unit	*Temperature (°C)*	*Seafood**	*log (count)*
1	0	1	3.6882
1	0	2	0.3565
2	0	1	1.8275
2	0	2	1.7023
3	0	1	5.2327
3	0	2	4.5780
4	5	1	7.1950
4	5	2	5.0169
5	5	1	9.3224
5	5	2	7.9519
6	5	1	7.4195
6	5	2	6.3861
7	10	1	9.7842
7	10	2	10.1352
8	10	1	6.4703
8	10	2	5.0482
9	10	1	9.4442
9	10	2	11.0329

*1 = oysters, 2 = mussels

7. A split-plot experiment in a randomized complete block design evaluated the effects of nitrogen, water, and phosphorus rates on the water use efficiency in a drip irrigation culture of sweet corn. Two rates of phosphorus ($P_1 = 0$ and $P_2 = 245$ lb P_2O_5/acre) were randomized to whole plots in a randomized complete block design. The 3×3 factorial treatments of nitrogen (0, 130, and 260 lb N/acre) and water (16, 22, and 28 inches) were randomized to subplots within each of the main plots. The data shown are water use efficiency for each subplot.
   a. Write a linear model for the experiment, explain the terms, and conduct the analysis of variance for the data.
   b. Construct a summary table of cell means and marginal means for this experiment, and compute the estimated standard errors for the table of means.
   c. Compute the estimated standard errors for the differences between two observed means:
      (i) for phosphorus rates
      (ii) for water levels
      (iii) for nitrogen rates
      (iv) for water by nitrogen cell means
   d. Test the hypotheses for all interactions and main effects, and interpret the results.
   e. Partition the sum of squares for water and nitrogen into linear and quadratic polynomial regression partitions including interaction. Interpret the results utilizing a graph of the observed means to assist in the interpretation.

		Block 1		*Block 2*	
Water	*Nitrogen*	P_1	P_2	P_1	P_2
16	0	8.1	9.7	8.6	15.5
	130	36.0	34.2	34.5	33.1
	260	34.6	34.0	40.7	39.3
22	0	10.0	6.2	5.1	10.9
	130	21.5	19.7	19.9	21.9
	260	30.7	28.9	26.4	25.7
28	0	10.6	6.3	4.5	10.4
	130	19.4	19.7	21.7	19.9
	260	23.2	23.0	19.4	23.2

Source: Dr. T. Doerge, Department of Soil and Water Science, University of Arizona.